8. [10 points] For $\alpha > 0$, consider the family of spirals given by $r = \frac{1}{\theta^\alpha}$ in polar coordinates.

a. [2 points] Write down an integral that gives the length L of a spiral in this family for $1 \le \theta \le b$. No credit will be given if you just write down the formula given in part (b).

Solution: Using parametric equations: $x = \frac{1}{\theta^\alpha}\cos\alpha$ and $y = \frac{1}{\theta^\alpha}\sin\alpha$.

$$L = \int_1^b \sqrt{\left(-\frac{\alpha}{\theta^{\alpha+1}}\cos\theta - \frac{1}{\theta^\alpha}\sin\theta\right)^2 + \left(-\frac{\alpha}{\theta^{\alpha+1}}\sin\theta + \frac{1}{\theta^\alpha}\cos\theta\right)^2}\,d\theta$$

or

$$L = \int_1^b \sqrt{\left(\frac{1}{\theta^\alpha}\right)^2 + \left(-\frac{\alpha}{\theta^{\alpha+1}}\right)^2}\,d\theta$$

b. [8 points] It can be shown that the length L of the spiral in part a) may also be written as

$$L = \int_1^b \frac{1}{\theta^\alpha}\sqrt{1 + \frac{\alpha^2}{\theta^2}}\,d\theta.$$

Use this formula for L to find all values of $\alpha > 0$ for which the length of the spiral is infinite for $1 \le \theta$. For which values of α is the length finite? Justify all your answers using the comparison test.

Solution:

$$\int_1^\infty \frac{1}{\theta^\alpha}\sqrt{1 + \frac{\alpha^2}{\theta^2}}\,d\theta \qquad \text{behaves as} \qquad \int_1^\infty \frac{1}{\theta^\alpha}\,d\theta$$

Since

$$\int_1^\infty \frac{1}{\theta^\alpha}\,d\theta \le \int_1^\infty \frac{1}{\theta^\alpha}\sqrt{1 + \frac{\alpha^2}{\theta^2}}\,d\theta$$

then arc length is infinite of $\alpha \le 1$.

$$\int_1^\infty \frac{1}{\theta^\alpha}\sqrt{1 + \frac{\alpha^2}{\theta^2}}\,d\theta \le \int_1^\infty \frac{1}{\theta^\alpha}\sqrt{1+\alpha^2}\,d\theta = \sqrt{1+\alpha^2}\int_1^\infty \frac{1}{\theta^\alpha}\,d\theta$$

Hence the arc length is finite of $\alpha > 1$.