6. [13 points] A particle moves along the path given by the parametric equations

\[ x(t) = a \cos t \quad y(t) = \sin 2t \quad \text{for} \ 0 \leq t \leq 2\pi. \]

where \( a \) is a positive constant. The graph of the particle’s path in the \( x-y \) plane is shown below. In the questions below, show all your work to receive full credit.

a. [2 points] At which values of \( 0 \leq t \leq 2\pi \), does the particle pass through the origin?

Solution: \( 0 = x(t) = a \cos t \): \( t = \frac{\pi}{2}, \frac{3\pi}{2} \), \( 0 = y(t) = \sin 2t \): \( 2t = 0, \pi, 2\pi, 3\pi, 4\pi \) \( \Rightarrow t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi \). Particle passes through origin: \( t = \frac{\pi}{2}, \frac{3\pi}{2} \).

b. [5 points] For what values of \( a \) are the two tangent lines to the curve at the origin perpendicular? Hint: Two lines are perpendicular if the product of their slopes is equal to \(-1\).

Solution: \( x'(t) = -a \sin t, \ y'(t) = 2 \cos 2t. \)

\[
\begin{array}{c|c}
\frac{dx}{dt} & \frac{dx}{dt} = \frac{3\pi}{2} \\
\frac{dy}{dt} & \frac{dy}{dt} = -2 \\
\frac{dy}{dt} & \frac{dy}{dt} = \frac{a}{2} \\
\frac{dy}{dt} & \frac{dy}{dt} = -\frac{a}{2} \\
\end{array}
\]

\(-1 = \frac{a}{2} \left( -\frac{2}{a} \right) = -\frac{4}{a^2} \Rightarrow a = 2.\)

c. [4 points] At what values of \( 0 \leq t \leq 2\pi \), does the curve have horizontal tangents?

Solution: \( 0 = y'(t) = 2 \cos 2t \Rightarrow 2t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots \Rightarrow t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}. \)

d. [2 points] Find an expression that computes the length of the curve.

Solution: \( \int_{0}^{2\pi} \sqrt{a^2 \sin^2 t + 4 \cos^2 2t} dt. \)