7. [13 points] Consider the following improper integrals. Show all your work to receive full credit.

a. [5 points] Determine the convergence or divergence of the following improper integral. If the integral converges, compute its value.

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{\sin x}} dx$$

Solution: The integral is improper at $x = 0$ since $\sin 0 = 0$. Changing to a limit of proper integrals and using the substitution $u = \sin x$:

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{\sin x}} dx &= \lim_{a \to 0^+} \int_a^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{\sin x}} dx \\ &= \lim_{a \to 0^+} \int_{\sin a}^{1} u^{-1/2} du \\ &= \lim_{a \to 0^+} 2u^{1/2}\Big|_{\sin a}^{1} \\ &= \lim_{a \to 0^+} 2 - 2\sqrt{\sin a} \\ &= 2. \end{aligned}$$

Determine the convergence or divergence of the following improper integrals. Circle your answers.

b. [4 points] $\displaystyle\int_2^{\infty} \frac{5 - 3\sin(2x)}{x^2} dx$ **[Converges]** Diverges

Solution: Since $0 \le 5 - 3\sin(2x) \le 8$,

$$\int_2^{\infty} \frac{5 - 3\sin(2x)}{x^2} dx \le 8 \int_2^{\infty} \frac{1}{x^2} dx,$$

which converges by the p-test with $p = 2$.

c. [4 points] $\displaystyle\int_1^{\infty} \frac{1}{x}\sqrt{a^2 + \frac{1}{\sqrt{x}}}\, dx$, where a is a positive constant.

Converges **[Diverges]**

Solution: Since $a^2 + \frac{1}{\sqrt{x}} \ge a^2$ for $x > 0$,

$$\frac{1}{x}\sqrt{a^2 + \frac{1}{\sqrt{x}}} \ge \frac{a}{x},$$

and so

$$\int_1^{\infty} \frac{1}{x}\sqrt{a^2 + \frac{1}{\sqrt{x}}} dx \ge a \int_1^{\infty} \frac{1}{x} dx,$$

which diverges by the p-test, with $p = 1$.