8. [14 points] A coffee shop offers only one hour of free internet access to all its customers. The time \( t \) in hours a customer uses the internet at the coffee shop has a probability density function

\[
p(t) = \begin{cases} 
    at\sqrt{1-t^2} & 0 \leq t \leq 1, \\
    0 & \text{otherwise}.
\end{cases}
\]

where \( a \) is a constant.

a. [4 points] For what value of \( a \) is \( p(t) \) a probability density function? Find its value without using your calculator.

\[
\text{Solution:} \quad 1 = \int_0^1 at\sqrt{1-t^2} dt = \frac{a}{2} \int_0^1 \sqrt{u} du = \frac{a}{2} \left[ \frac{2}{3} u^{3/2} \right]_0^1 = \frac{a}{2} \left( \frac{2}{3} \right) = \frac{a}{3}.
\]

So, \( a = 3 \).

b. [4 points] Find the cumulative distribution function \( P(t) \) of \( p(t) \). Make sure to indicate the value of \( P(t) \) for all values of \(-\infty < t < \infty\). Your final answer should not contain any integrals.

\[
\text{Solution:} \quad P(t) = \int_{-\infty}^t p(x) dx, \text{ so if } t \leq 0 \text{ then } P(t) = 0, \text{ if } t \geq 1 \text{ then } P(t) = 1. \text{ If } 0 < t < 1,
\]

\[
P(t) = \int_0^t 3x\sqrt{1-x^2} dx = \frac{3}{2} \int_0^{1-t^2} \sqrt{u} du = -u^{3/2} \bigg|_0^{1-t^2} = 1 - (1-t^2)^{3/2}.
\]
c. [3 points] Find the probability that a customer is still using the internet after 40 minutes (without using your calculator).

Solution: The probability that a customer users the internet for 40 minutes or less is \( P(40/60) = P(2/3) \). So the probability of using the internet for more than 40 minutes is

\[
1 - P(2/3) = 1 - \left(1 - (1 - (2/3)^2)^{3/2}\right) = \left(1 - \frac{4}{9}\right)^{3/2} = \frac{\sqrt{125}}{27}.
\]


d. [3 points] Find an expression for the mean of this distribution. Use your calculator to compute its value.

Solution:

\[
\int_0^1 3t^2 \sqrt{1 - t^2} dt \approx 0.589 \text{ hours}.
\]