8. [14 points] A coffee shop offers only one hour of free internet access to all its customers. The time t in hours a customer uses the internet at the coffee shop has a probability density function

$$p(t) = \begin{cases} at\sqrt{1-t^2} & 0 \le t \le 1. \\ 0 & \text{otherwise.} \end{cases}$$

where a is a constant.

a. [4 points] For what value of a is $p(t)$ a probability density function? Find its value without using your calculator.

Solution:

$$1 = \int_0^1 at\sqrt{1-t^2}dt = -\frac{a}{2}\int_1^0 \sqrt{u}du = -\frac{a}{2}\frac{2}{3}u^{3/2}\bigg|_1^0 = -\frac{a}{2}\left(-\frac{2}{3}\right) = \frac{a}{3}.$$

So, $a = 3$.

b. [4 points] Find the cumulative distribution function $P(t)$ of $p(t)$. Make sure to indicate the value of $P(t)$ for all values of $-\infty < t < \infty$. Your final answer should not contain any integrals.

Solution: $P(t) = \int_{-\infty}^t p(x)dx$, so if $t \le 0$ then $P(t) = 0$, if $t \ge 1$ then $P(t) = 1$. If $0 < t < 1$,

$$P(t) = \int_0^t 3x\sqrt{1-x^2}dx = -\frac{3}{2}\int_1^{1-t^2} \sqrt{u}du = -\,u^{3/2}\bigg|_1^{1-t^2} = 1 - (1-t^2)^{3/2}.$$

c. [3 points] Find the the probability that a customer is still using the internet after 40 minutes (without using your calculator).

Solution: The probability that a customer users the internet for 40 minutes or less is $P(40/60) = P(2/3)$. So the probability of using the internet for more than 40 minutes is

$$1 - P(2/3) = 1 - \left(1 - (1 - (2/3)^2)^{3/2}\right) = \left(1 - \frac{4}{9}\right)^{3/2} = \frac{\sqrt{125}}{27}.$$

d. [3 points] Find an expression for the mean of this distribution. Use your calculator to compute its value.

Solution:

$$\int_0^1 3t^2\sqrt{1-t^2}dt \approx 0.589 \text{ hours}.$$