

4. [12 points] A volcano has erupted, covering its surroundings with ash. The area covered with ash is bounded by the polar curve $r = \frac{2}{2 + \sin(\theta + \frac{\pi}{4})}$, where the volcano is located at the origin and r is measured in km (see the shaded region in Figure 1).

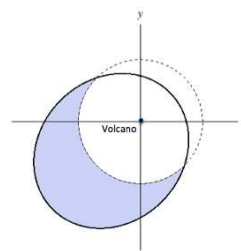
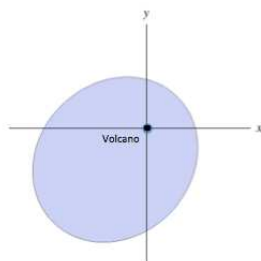


Figure 1. Affected area. Figure 2. Area to be cleaned.

- a. [3 points] Local authorities want to estimate the area of the region covered with ash. Write an integral that computes the total area covered with ash. You do not need to compute the value of the integral.

Solution:

$$A = \int_0^{2\pi} \frac{1}{2} \left(\frac{2}{2 + \sin(\theta + \frac{\pi}{4})} \right)^2 d\theta.$$

- b. [7 points] Days after the eruption, the government designs a plan to remove the ashes **outside** a circle of radius one km around the volcano (see the shaded region in Figure 2). Find an expression containing definite integrals that computes the cost of cleaning this region, assuming it costs 3 million dollars to clean the ashes in one km square of land. You do not need to compute the value of the integral.

Solution: The curves intersect when θ satisfies $1 = \frac{2}{2 + \sin(\theta + \frac{\pi}{4})}$. This equation leads to $\sin(\theta + \frac{\pi}{4}) = 0$. Then $\theta = \frac{3}{4}\pi$ and $\theta = \frac{7}{4}\pi$.

$$\text{Cost} = \$3 \text{ million} \cdot \left(\int_{\frac{3}{4}\pi}^{\frac{7}{4}\pi} \frac{1}{2} \left(\frac{2}{2 + \sin(\theta + \frac{\pi}{4})} \right)^2 d\theta - \int_{\frac{3}{4}\pi}^{\frac{7}{4}\pi} \frac{1}{2} d\theta \right).$$

or

$$\text{Cost} = \$3 \text{ million} \cdot \left(\int_{\frac{3}{4}\pi}^{\frac{7}{4}\pi} \frac{1}{2} \left(\frac{2}{2 + \sin(\theta + \frac{\pi}{4})} \right)^2 d\theta - \frac{\pi}{2} \right).$$

- c. [2 points] How far from the volcano is the farthest point covered with ash? Justify your answer.

Solution: Since $r = \frac{2}{2 + \sin(\theta + \frac{\pi}{4})}$ and $-1 \leq \sin(\theta + \frac{\pi}{4}) \leq 1$, then $\frac{2}{3} = \frac{2}{2+1} \leq \frac{2}{2 + \sin(\theta + \frac{\pi}{4})} \leq \frac{2}{2-1} = 2$. Hence the the farthest point covered with ash is 2 km away from the volcano.