9. [7 points]

Thanks to the Math Department's acquisition of a coffee tank in October, there are now 300 cups of coffee available to the graduate students each day.

The department wants to assess how much of the coffee is drunk and how much is wasted. Let c be the amount of coffee drunk in one day, measured in hundreds of cups of coffee. The probability density function for c is given by

$$p(c) = \begin{cases} \frac{3}{20}c^2 & \text{for } 0 \le c \le 2 \\ \frac{3}{5} & \text{for } 2 \le c \le 3 \\ 0 & \text{otherwise.} \end{cases}$$
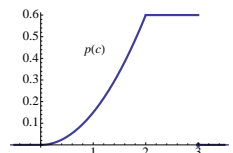
a. [4 points] Find the mean of the amount of coffee drunk in one day. Include units. Show all your work.

Solution: The mean is

$$\int_{-\infty}^{\infty} c\,p(c)dc = \int_0^3 c\,p(c)dc.$$

Since $p(c)$ is defined piecewise, we break the integral into the two pieces:

$$\int_0^3 cp(c)dc = \int_0^2 c\cdot\tfrac{3}{20}c^2 dc + \int_2^3 c\cdot\tfrac{3}{5}dc$$

$$= \tfrac{3}{80}c^4 \Big|_0^2 + \tfrac{3}{10}c^2\Big|_2^3 = \tfrac{3}{80}\cdot(16-0) + \tfrac{3}{10}(9-4)$$

$$= 2.1 \text{ or } \tfrac{168}{80}.$$

So, the mean amount of coffee drunk in one day is **210 cups of coffee**.

b. [3 points] Find the median of the amount of coffee drunk in one day. Include units. Show all your work.

Solution: We want to find M such that

$$\int_0^M p(c)dc = \int_M^3 p(c)dc = \frac{1}{2}.$$

From the graph, we see that the area between $c=2$ and $c=3$ is $\frac{3}{5}$ (since it is rectangular), which is already more than $\frac{1}{2}$. So, the median will be in the interval $2 \le c \le 3$, and we can use the second part of the piecewise formula:

$$\tfrac{1}{2} = \int_M^3 \tfrac{3}{5}dc = \tfrac{3}{5}(3-M),$$

so $M = 3 - \frac{5}{6} \approx 2.17$. So, the median amount of coffee drunk in one day is **217 cups**.

(**Note**: the area in the first interval is $\int_0^2 \tfrac{3}{20}c^2 dc = \tfrac{1}{20}\cdot 2^3 = \tfrac{2}{5}$, which is 0.1 less than $\frac{1}{2}$. So, we could instead solve $\int_2^M \tfrac{3}{5}dc = 0.1$.)