3. [16 points]

a. [5 points] Kazilla uses the ingredients you acquired from the store to make a special potion. Kazilla starts with a bucket containing 5 L of water and steadily adds purple liquid to the bucket at a rate of 0.5 L/min. Once in the bucket, the purple liquid mixes instantly with the water and the mixture drains out of the bottom of the bucket at a rate of 0.5 L/min. Let $P(t)$ give the amount of purple liquid in the bucket t minutes after Kazilla starts making the potion. Write a differential equation involving $P(t)$.

Solution:

$$\frac{dP}{dt} = 0.5 - \frac{P}{10}$$

b. [7 points] Later, Kazilla asks you to find the correct amount of green liquid to add to her potion. Let $G(t)$ be the total amount of green liquid you need to add, in liters, after t minutes. Suppose $G(1) = \frac{3}{2}$ and $G(t)$ satisfies the differential equation

$$\frac{dG}{dt} = -2t(G-1)^2.$$

Find $G(t)$.

Solution: Start with a separation of variables:

$$\int \frac{dG}{(G-1)^2} = \int -2t\,dt$$

$$\frac{-1}{(G-1)} = -t^2 + C$$

$$G = \frac{1}{t^2 + C} + 1$$

Using the initial condition that $G(1) = 3/2$, we can solve for C:

$$\frac{3}{2} = \frac{1}{1+C} + 1$$

$$C = 1$$

So $G(t) = \dfrac{1}{t^2+1} + 1$

c. [4 points] Finally, Kazilla asks you to add an amount of blue liquid $B(t)$ at a rate given by

$$\frac{dB}{dt} = -2B + e^{t/2}.$$

For which value(s) of c is the function

$$B(t) = 7e^{-2t} + \frac{ce^{t/2}}{5}$$

a solution to this differential equation?

Solution:

$$B'(t) = -14e^{-2t} + \frac{ce^{t/2}}{10}$$

$$B'(t) = \frac{dB}{dt} = -2(7e^{-2t} + \frac{ce^{t/2}}{5}) + e^{t/2} = -14e^{-2t} + (1 - \frac{2c}{5})e^{t/2}$$

So $1 - \frac{2c}{5} = \frac{c}{10}$ or $c = 2$.