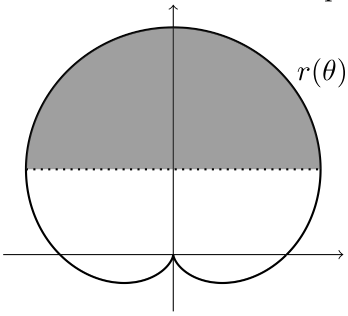
8. [11 points] Franklin, your robot, uses the lasers on his satellites to burn strange shapes in local corn fields. One of these strange shapes is given by the polar equation $r(\theta) = 2 + 2\sin(\theta)$ where $r(\theta)$ is measured in kilometers. All of the corn **above** the line $y = \frac{3}{2}$ has been pecked away by a flock of wild chickens. The polar curve $r(\theta)$ (solid) and the line $y = \frac{3}{2}$ (dotted) are shown below. The portion of the corn field that has been pecked away is shaded below.



a. [6 points] Write an expression involving one or more integrals which gives the area of the shaded region. Do not evaluate any integrals. **Include units.**

Solution: First, we need to find the values of θ where $r(\theta)$ intersects the line $y = 3/2$. Since $y = r\sin(\theta)$, we have that

$$3/2 = [2 + 2\sin(\theta)]\sin(\theta)$$

This is a quadratic equation in $\sin(\theta)$, and the quadratic formula gives $\sin(\theta) = 1/2$ or $-3/2$. Since $\sin(\theta)$ is bounded between -1 and 1, $\sin(\theta) = 1/2$. This means that:

$$\theta = \pi/6 \qquad \text{or} \qquad 5\pi/6$$

The line $y = 3/2$ has polar equation $r_{\text{line}}(\theta) = \frac{3}{2\sin(\theta)}$. Therefore the area bounded between $r(\theta)$ and $r_{\text{line}}(\theta)$ is given by the integral:

$$\frac{1}{2}\int_{\pi/6}^{5\pi/6} r(\theta)^2 - r_{\text{line}}(\theta)^2 d\theta = \frac{1}{2}\int_{\pi/6}^{5\pi/6} (2 + 2\sin(\theta))^2 - \left(\frac{3}{2\sin(\theta)}\right)^2 d\theta \qquad \text{km}^2$$

b. [5 points] Write an expression involving one or more integrals which gives the perimeter of the shaded region. Do not evaluate any integrals. **Include units.**

Solution: To find the length of the portion of the line $y = 3/2$ passing through the shaded region, we can take the difference of the x-coordinates of the points where $r(\theta)$ intersects the line. Since $r_{\text{line}}(\theta) = \frac{3}{2\sin(\theta)}$, the x-coordinates are given by $x = r_{\text{line}}(\theta)\cos(\theta) = \frac{3\cos(\theta)}{2\sin(\theta)}$

$$\text{Length of segment} = \frac{3\cos(\pi/6)}{2\sin(\pi/6)} - \frac{3\cos(5\pi/6)}{2\sin(5\pi/6)} \approx 5.196 \qquad \text{km}$$

Now we need to calculate the portion of the perimeter lying on $r(\theta)$. For this we can use the polar perimeter formula

$$\text{Length over } r(\theta) = \int_{\pi/6}^{5\pi/6} \sqrt{(2 + 2sin(\theta))^2 + (2\cos(\theta))^2} d\theta \quad \text{km}$$

$$\text{Perimeter} = \frac{3\cos(\pi/6)}{2\sin(\pi/6)} - \frac{3\cos(5\pi/6)}{2\sin(5\pi/6)} + \int_{\pi/6}^{5\pi/6} \sqrt{(2 + 2sin(\theta))^2 + (2\cos(\theta))^2} d\theta \quad \text{km}$$