9. [10 points] Determine whether the following integrals converge or diverge. Justify your answer.

a. [5 points] $\displaystyle\int_1^\infty \frac{x^2}{e^{-x}+3x^3}dx$

Solution:
We know that $e^{-x} < x^3$ for $x > 1$. Therefore

$$\int_1^\infty \frac{x^2}{e^{-x}+3x^3}dx > \int_1^\infty \frac{x^2}{4x^3}dx$$

We can simplify the integral on the right:

$$\int_1^\infty \frac{x^2}{4x^3}dx = \frac{1}{4}\int_1^\infty \frac{1}{x}dx$$

We know that the integral $\displaystyle\int_1^\infty \frac{1}{x}dx$ diverges, so this larger integral must also diverge.

b. [5 points] $\displaystyle\int_0^1 \frac{x\cos(x)-\sin(x)}{x^2}dx$

$$\left(\textit{Hint}: \quad \frac{d}{dx}\left(\frac{\sin(x)}{x}\right) = \frac{x\cos(x)-\sin(x)}{x^2}.\right)$$

Solution:

$$\begin{aligned}\int_0^1 \frac{x\cos(x)-\sin(x)}{x^2}dx &= \lim_{b\to 0^+}\int_b^1 \frac{x\cos(x)-\sin(x)}{x^2}dx \\ &= \lim_{b\to 0^+}\left.\frac{\sin(x)}{x}\right|_{x=b}^1 \\ &= \frac{\sin(1)}{1} - \lim_{b\to 0^+}\frac{\sin(b)}{b}\end{aligned}$$

To evaluate the limit, we need to use L'Hopital's Rule:

$$\lim_{b\to 0}\frac{\sin(b)}{b} = \lim_{b\to 0}\frac{\cos(b)}{1} = 1.$$

We can now give an exact value for the integral.

$$\int_0^1 \frac{x\cos(x)-\sin(x)}{x^2}dx = \frac{\sin(1)}{1} - 1 < \infty$$

and therefore the integral converges.