7. [18 points] A certain small country called Merrimead has 25 million dollars in paper currency in circulation, and each day 50 thousand dollars comes into Merrimead's banks. The government decides to introduce new currency by having the banks replace the old bills with new ones whenever old currency comes into the banks. Assume that the new bills are equally distributed throughout all paper currency. Let $M = M(t)$ denote the amount of new currency, in thousands of dollars, in circulation at time t days after starting to replace the paper currency.

a. [5 points] Write a differential equation involving $M(t)$, including an appropriate initial condition.

Solution: The concentration of old bills among all bills in circulation is $\frac{25000-M}{25000}$, and 50 thousand dollars moves through the bank each day, so

$$\frac{dM}{dt} = \text{Concentration} \times \text{Money per day} = \frac{25000 - M}{25000} \cdot 50, \qquad M(0) = 0.$$

Now consider the differential equation

$$B^2 + 2B\frac{dB}{dt} = 2500.$$

b. [4 points] Find all equilibrium solutions and classify their stability.

Solution: If $\frac{dB}{dt} = 0$ then we see that $B^2 = 2500$, so the equilibrium solutions are $B = \pm 50$. Both equilibrium solutions are stable.

Brightcrest, a second small country, also wants to replace all of their old paper bills as well, using a different strategy than Merrimead. The amount $B(t)$, in millions of dollars, of new paper currency in circulation in Brightcrest at a time t years after starting to replace the paper currency is modeled by the differential equation for B above with initial condition $B(0) = 0$.

c. [6 points] Find a formula for $B(t)$.

Solution: Using separation of variables, we have $\int \frac{2B}{2500-B^2}\, dB = \int dt$. Using the substitution $w = 2500 - B^2, dw = -2B\, dB$, we see that $-\ln|2500 - B^2| = t + C$. Solving for B, we see $B = \sqrt{2500 - Ae^{-t}}$.
Since $B(0) = 0$, we see that $A = 2500$, so

$$B(t) = \sqrt{2500 - 2500e^{-t}}.$$

d. [3 points] Assuming that all of the old bills are replaced in the long run, how much time will pass after starting to replace the paper bills until the new currency accounts for 99% of all currency in Brightcrest?

Solution: Since all of the bills are replaced in the long run, the total amount of money in circulation is $\lim_{t\to\infty} B(t) = 50$ million dollars. So the amount of time that passes until the new bills account for 99% of all currency is the value of t so that

$$(.99)(50) = \sqrt{2500 - 2500e^{-t}}.$$

So $t = -\ln(1 - (.99)^2) \approx 3.92$ years.