10. [9 points] The sequence $\{\gamma_n\}$ is defined according to the formula

$$\gamma_n = -\ln(n) + \sum_{k=1}^{n} \frac{1}{k}.$$

(You may recall this sequence from team homework 5.) This sequence converges to a positive number γ (which happens to be $\gamma \approx 0.5772156649$).

a. [2 points] Does the sequence $\{\gamma_n^2\}$ converge or diverge? If this sequence converges, compute the value to which this sequence converges, either in terms of the constant γ or with five decimal places of accuracy.

Solution: Yes, this sequence converges.

$$\lim_{n\to\infty} \gamma_n^2 = \left(\lim_{n\to\infty} \gamma_n\right)^2 = \gamma^2 \approx 0.33318.$$

b. [3 points] Does the series $\sum_{n=1}^{\infty} \gamma_n$ converge or diverge? Briefly explain your answer, and if this series converges, compute the value to which the series converges either in terms of the constant γ or with five decimal places of accuracy.

Solution: This series diverges by the n^{th} term test for divergence. The terms of this series do not approach 0; instead the terms approach γ, i.e.

$$\lim_{n\to\infty} \gamma_n = \gamma \approx 0.5772156649 \neq 0.$$

c. [4 points] Let $h_n = \sum_{k=1}^{n} \frac{1}{k}$. Find the value of $\lim_{n\to\infty} \frac{e^{h_n}}{n}$.

You may give your answer either in terms of the constant γ or with five decimal places of accuracy.

Hint: First consider $\lim_{n\to\infty} \ln\left(\frac{e^{h_n}}{n}\right)$.

Solution: Following the hint, we first compute

$$\ln\left(\frac{e^{h_n}}{n}\right) = \ln\left(\frac{e^{\sum_{k=1}^{n} 1/k}}{n}\right) = \ln(e^{\sum_{k=1}^{n} 1/k}) - \ln(n) = -\ln(n) + \sum_{k=1}^{n} \frac{1}{k} = \gamma_n.$$

Next, we take the limit of this expression.

$$\lim_{n\to\infty} \ln\left(\frac{e^{h_n}}{n}\right) = \lim_{n\to\infty} \gamma_n = \gamma \approx 0.5772156649.$$

Finally, the limit we are looking for is found by exponentiating this result (in order to "undo" the natural log from before).

$$\lim_{n\to\infty} \frac{e^{h_n}}{n} = e^{\gamma} \approx e^{0.5772156649} \approx 1.78107.$$