2. [9 points] Consider the graph of $y = e^{-14x}$ for $x \geq 0$.

a. [3 points] Let $\mathcal{R}$ be the region in the first quadrant between the graph of $y = e^{-14x}$ and the x-axis. Which of the following improper integrals best expresses the volume of the solid that is obtained by rotating $\mathcal{R}$ around the x-axis?

Circle one:

$\int_0^\infty \pi e^{-14x}\,dx$ $\int_0^\infty x e^{-14x}\,dx$ $\int_0^\infty x e^{-28x}\,dx$ $\boxed{\int_0^\infty \pi e^{-28x}\,dx}$

$\frac{\pi}{7}\int_0^1 \ln(y)\,dy$ $\frac{\pi}{14}\int_0^1 y\ln(y)\,dy$ $\frac{1}{14}\int_0^1 y\ln(y^2)\,dy$

Solution: A thin slice of this solid of thickness Δx taken perpendicular to the x-axis at the point x has approximate volume $\pi(e^{-14x})^2 = \pi e^{-28x}$ cubic units. If b is a positive number, the volume of the portion of this solid between $x = 0$ and $x = b$ is $\int_0^b \pi e^{-28x}\,dx$.

So the volume of the entire solid would be $\lim_{b\to\infty}\int_0^b \pi e^{-28x}\,dx = \int_0^\infty \pi e^{-28x}\,dx$.

b. [6 points] Determine whether the improper integral you circled in part **a** converges or diverges.

- If the integral converges, circle "converges", find its exact value (i.e. no decimal approximations), and write the exact value on the answer blank provided.
- If the integral diverges, circle "diverges" and justify your answer.

In either case, **you must show all your work carefully using correct notation**. Any direct evaluation of integrals must be done **without using a calculator.**

Converges to $\frac{\pi}{28}$ **Diverges**

Solution: We use the definition of this type of improper integral to find its value.

$$\begin{aligned}\int_0^\infty \pi e^{-28x}\,dx &= \lim_{b\to\infty}\int_0^b \pi e^{-28x}\,dx \\ &= \lim_{b\to\infty}\left[-\frac{\pi}{28}e^{-28x}\right]\Bigg|_{x=0}^{x=b} \\ &= \lim_{b\to\infty}\left[-\frac{\pi}{28}\left(e^{-28b}-1\right)\right] \\ &= \frac{\pi}{28}\ \text{(since } \lim_{b\to\infty} e^{-28b} = 0).\end{aligned}$$

So $\int_0^\infty \pi e^{-28x}\,dx$ converges to $\frac{\pi}{28}$.