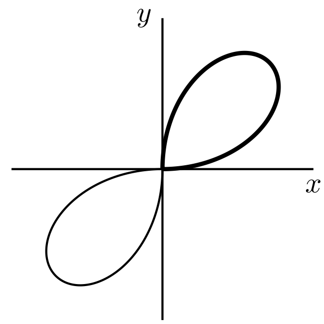
4. [13 points] The orbit of a single electron around the nucleus of an atom is determined by the energy level of that electron and by the other electrons orbiting the nucleus. We can model one electron's orbital in two-dimensions as follows. Suppose that the nucleus of an atom is centered at the origin. Then the (so-called "$2p_1$") orbital has the shape shown below.

This shape is made up of two regions that we call "lobes". The outer edge of the lobes are described by the polar equation $r = k\sin(2\theta)$ for some positive constant k. Note that only the relevant portion of the polar curve $r = k\sin(2\theta)$ is shown.

The "top lobe" is the portion in the first quadrant (shown in bold).



a. [2 points] For what values of θ with $0 \leq \theta \leq 2\pi$ does the polar curve $r = k\sin(2\theta)$ pass through the origin?

Solution: The curve passes through the origin when $r = 0$, i.e. when $\sin(2\theta) = 0$, or $\boxed{\theta = 0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2},\text{ and } 2\pi.}$

b. [3 points] For what values of θ does the polar curve $r = k\sin(2\theta)$ trace out the "top lobe"? Give your answer as an interval of θ values.

Solution: From part (a), we are at the origin when $\theta = 0$; as θ increases from 0, the radius $k\sin(2\theta)$ increases and then decreases (since k is positive). So we finish the lobe when we get to the second value of θ at which the curve intersects the origin, $\theta = \frac{\pi}{2}$. The final answer is thus $\boxed{0 \leq \theta \leq \pi/2 \text{ or, equivalently, the closed interval } [0, \pi/2]}$.

c. [4 points] Write, but do **not** evaluate, an integral that gives the area of the top lobe.

Solution: Using the formula for polar area and the values of θ from part ,

$$\text{area} = \int_0^{\pi/2} \frac{1}{2}(f(\theta))^2\, d\theta = \int_0^{\pi/2} \frac{(k\sin(2\theta))^2}{2}\, d\theta.$$

d. [4 points] Imagine that an electron lies within the top lobe of this orbital, but is as far away from the origin as possible. What are the polar coordinates of this point of greatest distance from the origin? Your answer may involve the constant k.

Solution: Maximizing distance from the origin means maximizing $|r|$, so want $\sin(2\theta) = \pm 1$. For θ in the interval $0 \leq \theta \leq \pi/2$, this implies that $2\theta = \pi/2$ so $\theta = \pi/4$. When $\theta = \pi/4$, we have $r = k\sin(\pi/2) = k$.

Therefore, the polar coordinates for this point are $\boxed{(r, \theta) = \left(k, \frac{\pi}{4}\right)}$.