

5. [5 points] Consider the improper integral $\int_1^{\infty} \frac{2}{3x + 5e^x} dx$.

Note that for $x > 0$, we have $\frac{2}{3x + 5e^x} < \frac{2}{3x}$ and $\frac{2}{3x + 5e^x} < 0.4e^{-x}$.

Use this information together with the (Direct) Comparison Test for Integrals to determine whether $\int_1^{\infty} \frac{2}{3x + 5e^x} dx$ converges or diverges.

Write the comparison function you use on the blank below and circle your conclusion for the improper integral. Then briefly explain your reasoning.

Answer: Using (direct) comparison of $\frac{2}{3x + 5e^x}$ with the function $0.4e^{-x}$,
the improper integral $\int_1^{\infty} \frac{2}{3x + 5e^x} dx$ Converges Diverges

Briefly explain your reasoning.

Solution: Note that the improper integral $\int_1^{\infty} e^{-x} dx$ is one of the “useful integrals for comparison” from the textbook. This integral is known to converge (exponential decay), so the improper integral $\int_1^{\infty} 0.4e^{-x} dx = 0.4 \int_1^{\infty} e^{-x} dx$ also converges. Together with the given inequality $\frac{2}{3x + 5e^x} < 0.4e^{-x}$ this implies that the improper integral $\int_1^{\infty} \frac{2}{3x + 5e^x} dx$ must also converge by the (Direct) Comparison Test for Improper Integrals.

6. [7 points] Consider the series $\sum_{n=2}^{\infty} \frac{n^2 - n + 2}{4n^4 - 3n^2}$.

Use the Limit Comparison Test to determine whether this series converges or diverges.

Circle your answer (either “converges” or “diverges”) clearly.

The series $\sum_{n=2}^{\infty} \frac{n^2 - n + 2}{4n^4 - 3n^2}$ Converges Diverges

Give full evidence to support your answer below. Be sure to clearly state your choice of comparison series, show each step of any computation, and carefully justify your conclusions.

Solution: By considering the exponents in the numerator and the denominator of the general term of this series, we decide to compare this series to $\sum_{n=2}^{\infty} \frac{1}{4n^2}$.

$$\lim_{n \rightarrow \infty} \frac{(n^2 - n + 2)/(4n^4 - 3n^2)}{1/4n^2} = \lim_{n \rightarrow \infty} \frac{(4n^2) \cdot (n^2 - n + 2)}{4n^4 - 3n^2} = 1.$$

Since this limit exists and is non-zero, we can apply the Limit Comparison Test. So the original series and the series $\sum_{n=2}^{\infty} \frac{1}{4n^2}$ either both converge or both diverge. The series $\sum_{n=2}^{\infty} \frac{1}{4n^2}$ is $1/4$ times a p -series ($p = 2$) with the first term ($n = 1$) omitted. Omitting a single term and multiplying by a non-zero constant do not affect the convergence of a series, so since the p -series with $p = 2$ converges, so too will the series $\sum_{n=2}^{\infty} \frac{1}{4n^2}$ converge.

By the Limit Comparison Test, we can therefore conclude that the original series $\sum_{n=2}^{\infty} \frac{n^2 - n + 2}{4n^4 - 3n^2}$ must also converge.