7. [12 points] A bouncy ball is launched up 20 feet from the floor and then begins bouncing. Each time the ball bounces up from floor, it bounces up again to a height that is 60% the height of the previous bounce. (For example, when it bounces up from the floor after falling 20 ft, the ball will bounce up to a height of $0.6(20) = 12$ feet.)
Consider the following sequences, defined for $n \geq 1$:

- Let h_n be the height, in feet, to which the ball rises when the ball leaves the ground for the nth time. So $h_1 = 20$ and $h_2 = 12$
- Let f_n be the total distance, in feet, that the ball has traveled (both up and down) when it bounces on the ground for the nth time. For example, $f_1 = 40$ and $f_2 = 40 + 24 = 64$.

a. [2 points] Find the values of h_3 and f_3.

Solution: $h_3 = 0.6(12) = 7.2$ and $f_3 = 64 + 14.4 = 78.4.$

Answer: $h_3 =$ 7.2 and $f_3=$ 78.4

b. [6 points] Find a closed form expression for h_n and f_n.
("Closed form" here means that your answers should not include sigma notation or ellipses $(\cdots)$. Your answers should also **not** involve recursive formulas.)

Solution: $h_n = 0.6h_{n-1}$ is a recursive relationship that holds between the terms of the sequence h_n for $n > 1$, and this recursive formula means that h_n is a geometric sequence. The (constant) ratio of successive terms is equal to 0.6 and first term is $h_1 = 20$. So we see that $h_n = 20(0.6)^{n-1}$.

Note that the term f_n is twice the sum of the first n terms of the h_n sequence. (Twice because the bouncy ball travels both up and down.) We use the formula for a partial sum of a geometric series (i.e. a finite geometric series) to find

$$\begin{aligned} f_n &= 2(h_1 + h_2 + ... + h_n) = 2(20 + ... + 20(0.6)^{n-1}) \\ &= \frac{2(20)(1-(0.6)^n)}{1-0.6} = \frac{40(1-(0.6)^n)}{0.4} = 100(1-(0.6)^n). \end{aligned}$$

Answer: $h_n =$ $20 \cdot (0.6)^{n-1}$ and $f_n=$ $\frac{40(1-(0.6)^n)}{0.4} = 100(1-(0.6)^n)$

c. [4 points] Decide whether the given sequence or series converges or diverges. If it diverges, circle "diverges". If it converges, circle "converges" and write the value to which it converges in the blank.

i. The sequence f_n

Converges to 100 **Diverges**

Solution: The limit of the sequence f_n is

$$\lim_{n\to\infty} f_n = \lim_{n\to\infty} \frac{40(1-(0.6)^n)}{0.4} = \frac{40}{0.4} = 100.$$

Since this limit exists, the sequence f_n converges, and this computation shows that it converges to 100.

Alternatively, as we saw in part **b**, the sequence f_n is the sequence of partial sums of the geometric series $\sum_{k=1}^{\infty} 2h_k = \sum_{k=1}^{\infty} 40(0.6)^{k-1}$. Since $r = 0.6$ and $|0.6| < 1$, we know that this geometric series converges to $\frac{40}{1-0.6} = 100$. By definition of series convergence, this sum is the limit of the sequence of partial sums f_n, i.e. $\lim_{n\to\infty} f_n = 100$.

ii. The series $\sum_{n=1}^{\infty} h_n$

Converges to 50 **Diverges**

Solution: Next, we consider the series $\sum_{n=1}^{\infty} h_n$, which we know is geometric from part **b**. Since the common ratio between successive terms is 0.6, the series converges, and the formula for the sum of a convergent geometric series gives us

$$\sum_{n=1}^{\infty} h_n = \sum_{n=1}^{\infty} 20 \cdot (0.6)^{n-1} = \frac{20}{1-0.6} = 50,$$

Alternatively, since the sequence f_n is the sequence of partial sums of the series $\sum_{k=1}^{\infty} 2h_k$, we have $\sum_{n=1}^{\infty} h_n = \frac{1}{2} \lim_{n\to\infty} f_n = \frac{100}{2} = 50$.