9. [12 points] For each of parts **a** through **c** below, determine the radius of convergence of the power series. Show your work carefully.

a. [3 points] $\displaystyle\sum_{n=1}^{\infty} \frac{e}{n!}(x-1)^n$

Solution: We apply the Ratio Test.

$$\lim_{n\to\infty} \frac{|(e(x-1)^{n+1})/(n+1)!|}{|(e(x-1)^n)/n!|} = \lim_{n\to\infty} \frac{e}{e} \cdot \frac{n!}{(n+1)!} \cdot |x-1| = \lim_{n\to\infty} \frac{1}{n+1}|x-1| = 0.$$

This limit is always less than one, so, by the Ratio Test, this power series will converge for every value of x. Hence the radius of convergence is ∞.

Answer: radius of convergence = ∞

b. [3 points] $5(x+\pi) + 5\cdot 4(x+\pi)^2 + 5\cdot 9(x+\pi)^3 + 5\cdot 16(x+\pi)^4 + \cdots$

Solution: We apply the Ratio Test.

$$\lim_{n\to\infty} \frac{|5(n+1)^2(x+\pi)^{n+1}|}{|5n^2(x+\pi)^n|} = \lim_{n\to\infty} \frac{n+1}{n}|x+\pi| = |x+\pi|.$$

The Ratio Test guarantees convergence when this limit is less than one (and divergence when the limit is greater than one). Now $|x+\pi| < 1$ means x is within 1 unit of π (or $-\pi - 1 < x < -\pi + 1$), so the radius of convergence is 1.

Answer: radius of convergence = 1

c. [3 points] $\displaystyle\sum_{n=0}^{\infty} \frac{\pi}{8^n}(x+2)^{3n}$

Solution: Note that this series is geometric with first term π and ratio of successive terms $\frac{(x+2)^3}{8}$, so the series converges if and only if $\left|\frac{(x+2)^3}{8}\right| < 1$.
Alternatively, we can apply the Ratio Test:

$$\lim_{n\to\infty} \frac{|\pi(x+2)^{3(n+1)}/8^{n+1}|}{|\pi(x+2)^{3n}/8^n|} = \lim_{n\to\infty} \frac{8^n}{8^{n+1}}|x+2|^3 = \frac{|x+2|^3}{8}$$

The Ratio Test guarantees convergence when this limit is less than one (and divergence when it is greater than one).

Using either approach, we see that the power series converges when $\dfrac{|x+2|^3}{8} < 1$, i.e. when $|x+2| < 2$.

Answer: radius of convergence = 2

d. [3 points] Consider the power series $\sum_{j=0}^{\infty} C_j(x-5)^j$, where each C_j is a constant. Suppose this power series

- converges when $x = 2$ and
- diverges when $x = 12$.

Based on this information, which of the following values **could** be equal to the radius of convergence of the power series? Circle all possibilities from the list below.

0	1	2	$\boxed{3}$	$\boxed{4}$
$\boxed{5}$	$\boxed{6}$	$\boxed{7}$	8	9
10	11	12	NONE OF THESE	