6. [10 points] Consider the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n\,5^n}(x+3)^n.$$

a. [2 points] What is the center of the interval of convergence of this power series?

Answer: The center is at $x =$ -3

For parts b and c below, show every step of any calculations and fully justify your answer with careful reasoning.

b. [3 points] Find the radius of convergence of this power series.

Ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x+3)^{n+1}}{(n+1)\,5^{n+1}} \cdot \frac{n\,5^n}{(-1)^n(x+3)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}}{(-1)^n}\cdot\frac{(x+3)^{n+1}}{(x+3)^n}\cdot\frac{\cancel{n}^{\,1}}{\cancel{n+1}}\cdot\frac{5^n}{5^{n+1}}\right| = \frac{|x+3|}{5}$$

Converges when ratio <1, i.e.

$$\frac{|x+3|}{5} < 1 \Rightarrow |x+3| < 5$$

Answer: Radius of Convergence: 5

c. [5 points] Find the interval of convergence for this power series.

Converges if $|x+3|<5$ and diverges if $|x+3|>5$.

On the boundary:

$\underline{x+3=5} \Rightarrow x=2 \Rightarrow$ Series $= \sum \frac{(-1)^n}{n5^n}\cdot 5^n = \sum \frac{(-1)^n}{n}$

which converges by Alternating Series Test, since signs alternate, $|$terms$|$ decreases, and terms $\to 0$.

$\underline{x+3=-5} \Rightarrow x=-8 \Rightarrow$ series $= \sum \frac{(-1)^n}{n5^n}(-5)^n = \sum \frac{1}{n}$

which diverges by the p-test $(p=1)$.

Answer: Interval of Convergence: $(-8, 2]$