8. [10 points] Determine whether the following improper integrals converge or diverge. Show all of your work and indicate any theorems you used to conclude convergence or divergence of the integrals. Any direct evaluation of integrals must be done <u>without</u> using a calculator.

a. [5 points] $\displaystyle\int_3^\infty \frac{\ln(x)}{x^2}\,dx$

Circle one: $\boxed{\textbf{Converges}}$ **Diverges**

Justification:

Let $u = \ln x$, $v' = x^{-2}$

$u' = x^{-1}$ $\quad v = -x^{-1}$

$$\int_3^\infty \frac{\ln x}{x^2}dx = \lim_{b\to\infty}\int_3^b uv'\,dx = \lim_{b\to\infty}\left[uv\Big|_3^b - \int_3^b u'v\,dx\right]$$

$$= \lim_{b\to\infty}\left[-\frac{\ln x}{x}\Big|_3^b - \int_3^b -x^{-2}dx\right] = \lim_{b\to\infty}\left[-\frac{\ln b}{b} + \frac{\ln 3}{3} - x^{-1}\Big|_3^b\right]$$

$$= \lim_{b\to\infty} \frac{-\ln b}{b} + \frac{\ln 3}{3} - \frac{1}{b} + \frac{1}{3} = \frac{1+\ln 3}{3} - \lim_{b\to\infty}\frac{1+\ln b}{b}$$

By L'Hôpital's Rule, that final limit is $\lim_{b\to\infty}\frac{1/b}{1} = 0$.

So converges to $\frac{1+\ln 3}{3}$

b. [5 points] $\displaystyle\int_0^\infty \frac{3}{4x^2+5\sqrt{x}}\,dx$

Circle one: $\boxed{\textbf{Converges}}$ **Diverges**

Justification:

on $(0,1)$, $\frac{3}{4x^2+5\sqrt{x}} < \frac{3}{5\sqrt{x}} = \frac{3}{5}\cdot\frac{1}{x^{1/2}}$. $\int_0^1 \frac{3}{5}\cdot\frac{1}{x^{1/2}}dx$ converges by the p-test $(p=\frac{1}{2})$, so $\int_0^1 \frac{3}{4x^2+5\sqrt{x}}dx$ converges by comparison

on $(1,\infty)$, $\frac{3}{4x^2+5\sqrt{x}} < \frac{3}{4x^2} = \frac{3}{4}\cdot\frac{1}{x^2}$. $\int_1^\infty \frac{3}{4}\frac{1}{x^2}$ converges by the p-test $(p=2)$, so $\int_1^\infty \frac{3}{4x^2+5\sqrt{x}}$ Converges by Comparison.

$$\therefore \int_0^\infty \frac{3}{4x^2+5\sqrt{x}}dx = \int_0^1 + \int_1^\infty \text{ Converges.}$$