9. [9 points] Consider the function $f(x) = \begin{cases} \dfrac{\sin(x^2)}{x} & x \neq 0 \\ c & x = 0. \end{cases}$

a. [2 points] Find the value of c that makes the function $f(x)$ continuous at $x = 0$. Show your work carefully.

$$\lim_{x\to 0} \frac{\sin(x^2)}{x} \underset{\text{by L'Hôpital}}{=} \lim_{x\to 0} \frac{2x\cos(x^2)}{1} = 2\cdot 0\cdot 1 = 0$$

Answer: $c =$ 0

b. [2 points] Let b_n be the nth positive value of x such that $f(x) = 0$. Write a formula for b_n.

$$f(x) = 0 \Rightarrow \frac{\sin(x^2)}{x} = 0 \Rightarrow \sin(x^2) = 0 \Rightarrow x^2 = n\pi$$

for some integer n.

Answer: $b_n =$ $\sqrt{n\pi}$

For parts **c** and **d** below, let $a_n = \displaystyle\int_{b_{n-1}}^{b_n} f(x)\,dx$ for $n \geq 1$.

c. [2 points] Explain why a_n is an alternating sequence.

Because sin alternates from + to − between its zeroes:

$$a_n = \int_{\sqrt{(n-1)\pi}}^{\sqrt{n\pi}} \frac{\sin(x^2)}{x}\,dx \qquad \text{Let } w = x^2, \Rightarrow dw = 2x\,dx$$

$$= \int_{(n-1)\pi}^{n\pi} \frac{\sin(w)}{2w}\,dw$$

d. [3 points] Compute $\lim\limits_{n\to\infty} a_n$. Provide clear justification and show your work.

$$|a_n| = \left| \int_{(n-1)\pi}^{n\pi} \frac{\sin(w)}{2w}\,dw \right| \leq \int_{(n-1)\pi}^{n\pi} \left|\frac{\sin(w)}{2w}\right| dw \leq \int_{(n-1)\pi}^{n\pi} \frac{dw}{2w}$$

$$= \frac{1}{2}\ln w \Big|_{(n-1)\pi}^{n\pi} = \frac{1}{2}\left[\ln(n\pi) - \ln((n-1)\pi)\right] = \frac{1}{2}\ln\left(\frac{n}{n-1}\right).$$

as $n\to\infty$, $\frac{n}{n-1} \to 1$, so $|a_n| \to 0$, so $a_n \to 0$

Answer: $\lim\limits_{n\to\infty} a_n =$ 0