5. [9 points]

a. [5 points] Determine the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{n!(3n)}{(2n)!\,3^n}(x-7)^n.$$

Show all your work.

Solution: Let $a_n = \frac{n!(3n)}{(2n)!\,3^n}(x-7)^n$. Then we will find the radius of convergence by applying the ratio test to $\sum_{n=0}^{\infty} a_n$.

$$\begin{aligned}
\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} &= \lim_{n\to\infty} \frac{\frac{(n+1)!(3n+3)}{(2n+2)!3^{n+1}}|x-7|^{n+1}}{\frac{n!(3n)}{(2n)!3^n}|x-7|^n} \\
&= \lim_{n\to\infty} \frac{(n+1)!(3n+3)(2n)!3^n|x-7|^{n+1}}{(2n+2)!3^{n+1}n!(3n)|x-7|^n} \\
&= \lim_{n\to\infty} \frac{(n+1)(3n+3)}{(2n+2)(2n+1)3(3n)}|x-7| \\
&= 0
\end{aligned}$$

for all x, since the degree of the denominator is greater than the degree of the numerator. This means that $\sum_{n=0}^{\infty} a_n$ converges for all x, and the radius is infinity.

Radius: ∞

b. [4 points] The power series $\sum_{n=0}^{\infty} \frac{(-1)^n}{6^n\sqrt{n^2+n+7}}(x-4)^n$ has radius of convergence $R = 6$.

At which of the following x-values does the power series converge? Circle <u>all</u> correct answers. You do not need to justify your answer.

i. $x = -6$

ii. $x = -2$

iii. $\boxed{x = 0}$

iv. $\boxed{x = 4}$

v. $\boxed{x = 6}$

vi. $\boxed{x = 10}$

vii. $x = 12$

viii. NONE OF THESE