8. [8 points] Determine whether the following series converges absolutely, converges conditionally, or diverges. Be sure to fully justify your answer, showing all work and indicating any theorems you use.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n+\sqrt{n^2+1}}$$

(Circle one): **Converges Absolutely** **Converges Conditionally** **Diverges**

Solution: We have $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n+\sqrt{n^2+1}} \right| = \sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n^2+1}}$. Notice

$$\lim_{n\to\infty} \frac{\frac{1}{n+\sqrt{n^2+1}}}{\frac{1}{n}} = \frac{1}{2},$$

so since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges since it is a $p-$series, with $p = 1$, the series $\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n^2+1}}$ diverges by limit comparison test. This tells us that the series cannot converge absolutely, but it may still converge conditionally.

So we need to check convergence of the series without absolute value. Since this is an alternating series, $\lim_{n\to\infty} \frac{1}{n+\sqrt{n^2+1}} = 0$, and $\frac{1}{n+\sqrt{n^2+1}}$ is decreasing, the series satisfies the hypotheses of the Alternating Series Test, and therefore we conclude that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+\sqrt{n^2+1}}$ converges.