6. [10 points] The power series

$$\sum_{n=1}^{\infty} \frac{6}{7^n\sqrt{n^2+2n}}(x-3)^n$$

has radius of convergence 7. (You can assume this to be true, and you do NOT need to verify this).

Find the **interval** of convergence for the power series. Show all your work including full justification of convergence and divergence of any relevant series.

Solution: From the form of the power series, the interval must be centered at $x = 3$. Since the radius of convergence is 7, that means the endpoints are at $x = -4$ and $x = 10$.

At $x = -4$, the power series becomes

$$\sum_{n=1}^{\infty} \frac{6}{7^n\sqrt{n^2+2n}}(-7)^n = \sum_{n=1}^{\infty} \frac{6(-1)^n}{\sqrt{n^2+2n}}.$$

This is an alternating series, and for $a_n = \frac{6}{\sqrt{n^2+2n}}$, we have $0 < a_{n+1} < a_n$ and $\lim_{n\to\infty} a_n = 0$, and so by the Alternating Series Test, the power series converges at $x = -4$.

At $x = 10$, the power series becomes

$$\sum_{n=1}^{\infty} \frac{6}{7^n\sqrt{n^2+2n}}(7)^n = \sum_{n=1}^{\infty} \frac{6}{\sqrt{n^2+2n}}.$$

Now $\frac{6}{\sqrt{n^2+2n}} \geq \frac{6}{\sqrt{3n^2}}$ for $n \geq 1$, and

$$\sum_{n=1}^{\infty} \frac{6}{\sqrt{3n^2}} = \sum_{n=1}^{\infty} \frac{6}{n\sqrt{3}}$$

diverges by the p-test (with $p = 1$). So by the (Direct) Comparison Test,

$$\sum_{n=1}^{\infty} \frac{6}{\sqrt{n^2+2n}}$$

diverges. Therefore, the interval of convergence for the power series is $[-4, 10)$.

Answer: $[-4, 10)$