7. [16 points] Gabriel the aspiring jazz musician owns a number of cats and kittens which he lets wander his neighborhood. When he wants to feed them, he blows his trusty cat trumpet, and waits for them to come running.

a. [6 points] The probability density function for the time t, in minutes, that it takes for Miles the cat to arrive is given by $m(t)$ where

$$m(t) = \begin{cases} 0 & \text{for } t < 0 \\ \frac{1}{5} & \text{for } 0 \le t \le a \\ \frac{1}{5}e^{-t+a} & \text{for } t > a \end{cases}$$

for some constant a. Find the value of a.

Solution: Since $m(t)$ is a pdf, we must have $\displaystyle\int_{-\infty}^{\infty} m(t)\,dt = 1$, and so

$$\begin{aligned} \frac{1}{5}a + \int_a^{\infty} \frac{1}{5}e^{-t+a}\,dt &= 1 \\ \frac{1}{5}a + \lim_{b\to\infty} \int_a^b \frac{1}{5}e^{-t+a}\,dt &= 1 \\ \frac{1}{5}a - \lim_{b\to\infty} \frac{1}{5}e^{-t+a}\Big|_a^b\,dt &= 1 \\ \tfrac{1}{5}a + \tfrac{1}{5} &= 1 \end{aligned}$$

and so $a = 5 - 1 = 4$.

b. [7 points] Find the mean time in minutes that it takes Miles to arrive. You should evaluate any integrals or limits in your expression. You may give your answer in terms of a, but not in terms of m. You are not required to simplify your answer.

Solution: The mean time is given by

$$\begin{aligned} \int_{-\infty}^{\infty} tm(t)\,dt &= \int_0^a \frac{t}{5}\,dt + \int_a^{\infty} \frac{t}{5}e^{-t+a}\,dt \\ &= \frac{a^2}{10} + \frac{1}{5}\lim_{b\to\infty}\int_a^b te^{-t+a}\,dt \\ &= \frac{a^2}{10} + \frac{1}{5}\lim_{b\to\infty}\left(-te^{-t+a}\Big|_a^b + \int_a^b e^{-t+a}\,dt\right) \\ &= \frac{a^2}{10} + \frac{1}{5}\lim_{b\to\infty}\left(\left(-te^{-t+a} - e^{-t+a}\right)\Big|_a^b\right) \\ &= \frac{a^2}{10} + \frac{1}{5}(a+1) \end{aligned}$$

c. [3 points] The cumulative distribution function for the amount of time that it takes for Ella the kitten to arrive is given by $E(t)$. Gabriel knows that 18% of the time Ella arrives in less than 2 minutes, and that 40% of the time she takes more than 6 minutes to arrive. Use this information to find the value of $E(6) - E(2)$.

Solution: From the given information, we see that $E(2) = 0.18$ and $E(6) = 1 - 0.4 = 0.6$. Therefore $E(6) - E(2) = 0.6 - 0.18 = 0.42$.

Answer: **0.42**