3. [14 points] Molly has recently become a sheep herder. She rotates her sheep through various fields so that the sheep have a varied diet and the fields have a chance to grow. Every Monday, the sheep visit the same field. Before the sheep graze for the first time in this field, its grass is 20 centimeters tall. Molly's sheep are picky and only eat the top 40% of the length of grass in this field every Monday. Over the course of the week, before the next Monday, the grass grows 3 centimeters. Let G_i represent the height in centimeters of the grass right before the sheep graze on it for the ith time. Note that $G_1 = 20$.

a. [5 points] Find expressions for each of G_2, G_3, and G_4. You do not need to evaluate your expressions.

Solution:

$$
\begin{aligned}
G_2 &= (0.6)G_1 + 3 \\
&= (0.6)(20) + 3 \\
G_3 &= (0.6)G_2 + 3 \\
&= (0.6)^2(20) + (0.6)(3) + 3 \\
G_4 &= (0.6)G_3 + 3 \\
&= (0.6)^3(20) + (0.6)^2(3) + (0.6)(3) + 3
\end{aligned}
$$

b. [5 points] Find a general **closed-form** expression for G_n, defined for $n = 2, 3, 4 \ldots$

Solution:

$$
\begin{aligned}
G_n &= (0.6)^{n-1}(20) + \sum_{i=0}^{n-2} 3(0.6)^i \\
&= (0.6)^{n-1}(20) + \frac{3(1 - (0.6)^{n-1})}{1 - 0.6}
\end{aligned}
$$

c. [4 points] In order for the field to meet sheep grazing standards, the height of the grass must be at least 5 cm when the sheep begin grazing. Molly thinks she will be able to stay on her field forever. Help her determine whether she can stay by either showing that the grass will eventually be less than 5 cm in height, or showing that the grass will be at least 5 cm each time before the sheep graze.

Solution:

$$
\lim_{n \to \infty} G_n = \frac{3}{1 - 0.6} = 7.5.
$$

Also note that G_n is a decreasing sequence. So, the grass is always taller than 5 cm. when the sheep begin grazing.