9. [20 points] Otto would like to landscape his yard, so he contacts the company Granville's Calculate-Yourself Gardening. Granville's company provides each potential customer with a list of possible equations and improper integrals guiding the landscaping of the yard. Granville's company also offers a discount to customers that can correctly solve the integrals. Otto, strapped for cash, desperately wants to solve the equations Granville has sent him.

a. [3 points] Granville has also informed Otto that he can plant a maple tree in the back of the yard that is initially 2 meters tall. Granville estimates that the maple tree will grow at an instantaneous rate of

$$M(t) = \frac{12t}{e^t}$$

meters per year t years after it is planted. Write an integral that gives the height of the tree t years after it is planted. Your answer should not involve the letter M.

Solution:

$$2 + \int_0^t \frac{12s}{e^s} ds.$$

b. [7 points] Determine the maximum height that the maple tree will grow to.

Solution: Let us first calculate the integral in (a) using integration by parts ($u = 12s, dv = e^{-s}ds$):

$$\begin{aligned}\int_0^t \frac{12s}{e^s} ds &= -12se^{-s}|_0^t + 12\int_0^t e^{-s}ds \\ &= -12te^{-t} + 12\Big[-e^{-s}|_0^t\Big] \\ &= -12te^{-t} - 12e^{-t} + 12\end{aligned}$$

Since the tree is always growing, the maximum height is

$$\begin{aligned}2 + \int_0^\infty \frac{12t}{e^t} dt &= 2 + \lim_{t\to\infty} \int_0^t \frac{12s}{e^s} ds \\ &= 2 + \lim_{t\to\infty} 12 - 12e^{-t} - 12te^{-t} \\ &= 14 - \lim_{t\to\infty} \frac{12t}{e^t} \\ &\overset{\text{L'H}\frac{\infty}{\infty}}{=} 14 - \lim_{t\to\infty} \frac{12}{e^t} \\ &= 14.\end{aligned}$$

So, the maximum height of the tree is 12 meters.

9. (continued)

c. [10 points] Granville tells Otto that he can get a truck to come and dispense dirt into the yard for 2 hours. The instantaneous rate that the truck will dispense dirt t hours after the truck arrives is

$$D(t) = \frac{(\sin(t))^2}{t^{5/2}\sqrt{2-t}}$$

pounds per minute. Show that $\int_0^2 D(t)dt$ converges. Justify all of your work.

Hint 1: Try splitting this into 2 integrals, one from 0 to 1, and the other from 1 to 2.

Hint 2: You may want to use the fact that $\sin t \le t$ *for* $t \ge 0$.

Solution: Following the hint,

$$\int_0^2 D(t)dt = \int_0^1 D(t)dt + \int_1^2 D(t)dt.$$

Starting with the first integral, note that on $[0, 1]$,

$$D(t) \le \frac{t^2}{t^{5/2}} \le t^{-1/2}$$

By the p-test ($p = \frac{1}{2}$), $\int_0^1 \frac{1}{\sqrt{t}}dt$ converges, so by the comparison test, $\int_0^1 D(t)dt$ converges.

Next, we perform a change of variables on the second integral($w = 2 - t$):

$$\begin{aligned}\int_1^2 D(t)dt &= \lim_{b\to 2}\int_1^b D(t)dt \\ &= \lim_{b\to 2} -\int_1^{2-b} \frac{(\sin(2-w))^2}{(2-w)^{5/2}\sqrt{w}}dw \\ &= \int_0^1 \frac{(\sin(2-w))^2}{(2-w)^{5/2}\sqrt{w}}dw\end{aligned}$$

On the interval $[0, 1]$,

$$\frac{(\sin(2-w))^2}{(2-w)^{5/2}\sqrt{w}} \le \frac{1}{\sqrt{w}}$$

By the p-test ($p = \frac{1}{2}$), $\int_0^1 \frac{1}{\sqrt{w}}dw$ converges. So, by the comparison test, $\int_1^2 D(t)dt$ converges. As both parts converge, we have verified that the integral $\int_0^2 D(t)dt$ converges.