3. [15 points] Carlos and Nancy are catching a train that leaves at 4pm. They leave their apartment for the train station at 12pm. The amount of time t (in hours) that elapses between the time they leave their apartment and the time they arrive at the train station is described by the following **probability density function** (pdf) $h(t)$:

$$h(t) = \begin{cases} 0 & t \leq 3 \\ a(t-3) & 3 < t \leq 4 \\ \frac{1}{4}e^{4-t} & 4 < t < \infty. \end{cases}$$

a. [5 points] What is the probability they arrive late for their train (i.e., what is the probability they arrive at the train station after 4pm)? Be sure to show work for your calculations, and be sure to use proper notation.

Solution: The probability they arrive after 4pm is given by

$$\int_4^\infty h(t)\,dt.$$

Substituting the formula given for $h(t)$ on this interval yields the improper integral

$$\int_4^\infty \frac{1}{4}e^{4-t}\,dt.$$

We evaluate this:

$$\begin{aligned} \int_4^\infty \frac{1}{4}e^{4-t}\,dt &= \lim_{b\to\infty} \int_4^b \frac{1}{4}e^{4-t}\,dt \\ &= \lim_{b\to\infty} \Big(-\frac{1}{4}e^{4-t} \Big)\Big|_4^b \\ &= \lim_{b\to\infty} \big(\frac{1}{4} - \frac{1}{4}e^{4-b}\big) \\ &= \frac{1}{4}. \end{aligned}$$

So the answer is 1/4.

b. [4 points] Find the value of a so that $h(t)$ is a probability density function. Be sure to show work for any calculations.

Solution: Since $h(t)$ is a pdf, its integral from $-\infty$ to ∞ evaluates to 1. We already found $\displaystyle\int_4^\infty \frac{1}{4}e^{4-t}\,dt = 1/4$, so

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} h(t)\,dt \\ &= \int_3^4 a(t-3)\,dt + \int_4^\infty \frac{1}{4}e^{4-t}\,dt \\ &= \int_3^4 a(t-3)\,dt + \frac{1}{4}. \end{aligned}$$

We evaluate the first integral:

$$\begin{aligned} \int_3^4 a(t-3)\,dt &= \frac{at^2}{2}\Big|_3^4 - 3at\Big|_3^4 \\ &= 8a - \frac{9a}{2} - 3a \\ &= \frac{a}{2}. \end{aligned}$$

Therefore we have

$$\frac{a}{2} = 1 - \frac{1}{4} = \frac{3}{4},$$

so $a = 3/2$.

c. [3 points] Give a practical interpretation of the fact that $h(4.5) = 0.15$.
Note that the output value has been rounded to the nearest hundredth.

Solution: The probability that Carlos and Nancy arrive between 3:59 and 4:01 (this is a time interval of 2 minutes, or 1/30 hour) is approximately $\frac{1}{30} \cdot 0.15$.

d. [3 points] Write an expression involving one or more integrals that gives the mean amount of time it takes Nancy and Carlos to travel to the train station. The letter h should not appear in your answer. You do not need to evaluate any integrals for this part.

Solution: The mean is

$$\int_{-\infty}^{\infty} t h(t)\, dt = \int_{3}^{4} \frac{3}{2} t(t-3)\, dt + \int_{4}^{\infty} \frac{1}{4} e^{4-t}\, dt.$$