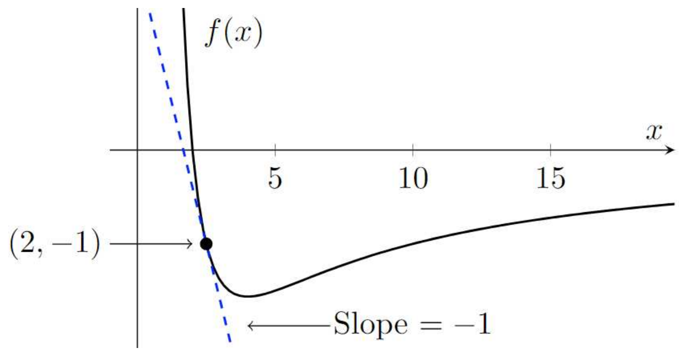
5. [12 points] The solid curve graphed below is part of the graph of a function $f(x)$ which has the following properties:

- $f(x)$ is twice differentiable on the interval $(0, \infty)$.
- $f(2) = -1$.
- For all $x \geq 10$, $f(x) < -\dfrac{5}{x}$.

The dashed line is the tangent line to $f(x)$ at $x = 2$, and its slope is -1.



a. [3 points] Compute $\displaystyle\lim_{x\to 2}\left(\frac{f(x)+1}{\cos\left(\frac{\pi}{2}x\right)+\frac{1}{2}x}\right)$

Solution: Note that since $f(x)$ is differentiable, hence continuous,

$$\lim_{x\to 2} f(x) + 1 = f(2) + 1 = 0.$$

Also, the limit of $\cos(\pi \cdot x/2) + x/2$ as $x \to 2$ is 0, so we apply L'Hôpital's rule:

$$\begin{aligned}\lim_{x\to 2}\frac{f(x)+1}{\cos\left(\frac{\pi}{2}x\right)+\frac{x}{2}} &\overset{LH}{=} \lim_{x\to 2}\frac{f'(x)}{-\frac{\pi}{2}\sin\left(\frac{\pi}{2}x\right)+\frac{1}{2}}\\ &= \frac{f'(2)}{1/2}\\ &= -2.\end{aligned}$$

b. [3 points] Compute $\displaystyle\lim_{x\to\infty} x[f(2+x^{-1})+1]$

Solution: Since the limit as x tends to infinity of $f(2+x^{-1})$ is $f(2) = -1$, this is an indeterminate limit of the form $\infty \cdot 0$, so L'Hôpital's rule applies. We get:

$$\begin{aligned}\lim_{x\to\infty} x[f(2+x^{-1})+1] &= \lim_{x\to\infty}\frac{f(2+x^{-1})+1}{x^{-1}}\\ &\overset{LH}{=} \lim_{x\to\infty}\frac{-x^{-2}f'(2+x^{-1})}{-x^{-2}}\\ &= f'(2)\\ &= -1.\end{aligned}$$

c. [6 points] Does the following improper integral converge or diverge? Fully justify your answer including using proper notation and showing mechanics of any tests you use.

$$\int_1^\infty (-f(x))\, dx.$$

Solution: This diverges by the comparison test. It suffices to check that $\int_{10}^\infty f(x)\, dx$ diverges. For this, we note that the inequality $f(x) < -5/x$ implies that

$$-f(x) > \frac{5}{x},$$

since $A < B$ implies that $-A > -B$. Furthermore, $\int_{10}^\infty \frac{5}{x}\, dx$ diverges by the p-test with $p = 1$. By the comparison test, $\int_{10}^\infty (-f(x))\, dx$ diverges as well.