5. [12 points] The solid curve graphed below is part of the graph of a function $f(x)$ which has the following properties:

- $f(x)$ is twice differentiable on the interval $(0, \infty)$.
- $f(2) = -1$.
- For all $x \geq 10$, $f(x) < -\frac{5}{x}$.

The dashed line is the tangent line to $f(x)$ at $x = 2$, and its slope is $-1$. 

\[ f(x) \]

(2, -1)

Slope = -1

a. [3 points] Compute $\lim_{x \to 2} \left( \frac{f(x) + 1}{\cos \left( \frac{\pi}{2} x \right) + \frac{1}{2} x} \right)$

\textbf{Solution:} Note that since $f(x)$ is differentiable, hence continuous,

$$\lim_{x \to 2} f(x) + 1 = f(2) + 1 = 0.$$ 

Also, the limit of $\cos(\pi \cdot x/2) + x/2$ as $x \to 2$ is $0$, so we apply L'Hôpital's rule:

$$\lim_{x \to 2} \frac{f(x) + 1}{\cos \left( \frac{\pi}{2} x \right) + \frac{1}{2} x} \overset{LH}{=} \lim_{x \to 2} \frac{f'(x)}{-\frac{\pi}{2} \sin \left( \frac{\pi}{2} x \right) + \frac{1}{2}}$$

$$= \frac{f'(2)}{1/2}$$

$$= -2.$$ 

b. [3 points] Compute $\lim_{x \to \infty} x[f(2 + x^{-1}) + 1]$

\textbf{Solution:} Since the limit as $x$ tends to infinity of $f(2 + x^{-1})$ is $f(2) = -1$, this is an indeterminate limit of the form $\infty \cdot 0$, so L'Hôpital’s rule applies. We get:

$$\lim_{x \to \infty} x[f(2 + x^{-1}) + 1] = \lim_{x \to \infty} \frac{f(2 + x^{-1}) + 1}{x^{-1}}$$

$$\overset{LH}{=} \lim_{x \to \infty} \frac{-x^{-2} f'(2 + x^{-1})}{-x^{-2}}$$

$$= f'(2)$$

$$= -1.$$
c. [6 points] Does the following improper integral converge or diverge? Fully justify your answer including using proper notation and showing mechanics of any tests you use.

\[ \int_{1}^{\infty} (-f(x)) \, dx. \]

**Solution:** This diverges by the comparison test. It suffices to check that \( \int_{10}^{\infty} f(x) \, dx \) diverges. For this, we note that the inequality \( f(x) < -5/x \) implies that

\[ -f(x) > \frac{5}{x}, \]

since \( A < B \) implies that \(-A > -B\). Furthermore, \( \int_{10}^{\infty} \frac{5}{x} \, dx \) diverges by the \( p \)-test with \( p = 1 \). By the comparison test, \( \int_{10}^{\infty} (-f(x)) \, dx \) diverges as well.