6. [11 points] It's fall, and the leaves are falling onto the grass. Over the course of each day, suppose that 10kg of leaves fall onto the grass. At the beginning of each day, 50 percent of the leaves currently on the grass (from previous days) are removed.

a. [3 points] Let M_n be the total mass in kg of all the leaves currently on the grass at the **end** of the nth day. Suppose that $M_1 = 10$. Write expressions for M_2 and M_3. The letter M should not appear in your answer.

Solution: $M_2 = \frac{1}{2} \cdot 10 + 10 = 15$, $M_3 = \frac{1}{2} \cdot 15 + 10 = 17.5$.

b. [5 points] Find a closed-form expression for M_n. This means that your answer should be a function of n, should not contain Σ, and should not be recursive.

Solution: Note that

$$\begin{aligned}
M_1 &= 10 \\
M_2 &= 10 + 10 \cdot \frac{1}{2} \\
M_3 &= 10 + \frac{1}{2}\left(10 + 10 \cdot \frac{1}{2}\right) = 10 + 10 \cdot \frac{1}{2} + 10 \cdot \left(\frac{1}{2}\right)^2 \\
&\vdots \\
M_n &= 10 + 10 \cdot \frac{1}{2} + 10 \cdot \left(\frac{1}{2}\right)^2 + \cdots + 10 \cdot \left(\frac{1}{2}\right)^{n-1} = \sum_{k=0}^{n-1} 10\left(\frac{1}{2}\right)^k.
\end{aligned}$$

Therefore

$$M_n = 10\frac{1-(1/2)^n}{1-1/2} = 20(1-(1/2)^n).$$

c. [3 points] If 30kg of leaves are on the grass at the end of any day, then the grass will die. Will this happen during the fall? Justify your answer.

Solution: The long-term mass in kg is found by taking the limit of our answer from part (b), and is

$$\lim_{n\to\infty} M_n = \lim_{n\to\infty} 10(1-(1/2)^n) = 20.$$

Since $20 < 30$, and since the sequence M_n is monotonically increasing, this means that $M_n \leq 20 < 30$ for all n. Therefore, there will never be 30 or more kg of leaves on the grass, so the grass will not die.