7. [7 points] Determine whether the following improper integral converges or diverges. Circle your final answer choice. Fully justify your answer including using proper notation and showing mechanics of any tests you use.

$$\int_1^\infty \frac{t^2 - \ln(t)}{t^4 + 8t + 10}\, dt.$$

Circle one: $\boxed{\textbf{Converges}}$ **Diverges**

Solution: The numerator is dominated by t^2, and the denominator is dominated by t^4, so the integrand has the same behavior (for large t) as $\frac{t^2}{t^4} = \frac{1}{t^2}$, whose integral on the interval $[1, \infty)$ converges. Therefore we expect that this improper integral converges. To show this, first note that since $\ln(t) \geq 0$ for $t \geq 1$, we have

$$t^2 - \ln(t) \leq t^2.$$

Also, since $8t + 10 \geq 0$ for $t \geq 1$, we have

$$t^4 + 8t + 10 \geq t^4.$$

Therefore

$$\frac{t^2 - \ln(t)}{t^4 + 8t + 10} \leq \frac{t^2}{t^4} = \frac{1}{t^2}.$$

Now, $\int_1^\infty \frac{1}{t^2}\, dt$ converges by the p-test with $p = 2$. Hence, by the comparison test, our integral converges as well.

8. [5 points] Fully evaluate the following integral:

$$\int x \ln x \, dx$$

You do not need to simplify your answer.

Solution: This integral can be done via integration by parts in two ways.
Method 1. Let $u = x, dv = \ln(x)\,dx$, so $du = dx, v = x\ln(x) - x$. Then integration by parts gives

$$\int x\ln(x)\,dx = -\int x\ln(x)\,dx + \int x\,dx + x(x\ln(x) - x).$$

Adding $\int x\ln(x)\,dx$ to both sides and performing the antidifferentiation on the right-hand side gives

$$2\int x\ln(x)\,dx = -\frac{1}{2}x^2 + x^2\ln(x) + C.$$

We then solve to get

$$\int x\ln(x)\,dx = \frac{1}{2}x^2\ln(x) - \frac{1}{4}x^2 + C.$$

Method 2. Let $u = \ln(x), dv = x\,dx$. Then $du = \frac{1}{x}\,dx$ and $v = \frac{x^2}{2}$. Integration by parts gives

$$\begin{aligned}\int x\ln(x)\,dx &= -\int \frac{x^2}{2}\cdot\frac{1}{x}\,dx + \ln(x)\cdot\frac{x^2}{2}\\ &= -\int \frac{1}{2}x\,dx + \frac{x^2}{2}\ln(x)\\ &= \frac{x^2}{2}\ln(x) - \frac{x^2}{4} + C.\end{aligned}$$