2. [7 points] Let \( f(x) \) and \( g(x) \) be two continuous and differentiable functions on \([1, \infty)\). Further, suppose these functions have the following properties:

- \( F(x) = \frac{g(x)}{x + \ln(x)} \) is an antiderivative of \( f(x) \) for \( x \geq 1 \),
- \( g(1) = 11 \),
- \( \lim_{x \to \infty} g(x) = \infty \),
- \( \lim_{x \to \infty} g'(x) = 21 \).

**Compute** the value of the following improper integral if it converges. If it does not converge, use a direct computation of the integral to show its divergence. Be sure to show your full computation, and be sure to use proper notation.

\[
\int_1^\infty f(x) \, dx
\]

*Circle one:* Diverges \[\text{Converges to } 10\]

**Solution:** We start by rewriting this improper integral as a limit, and then use the First Fundamental Theorem of Calculus:

\[
\int_1^\infty f(x) \, dx = \lim_{b \to \infty} \int_1^b f(x) \, dx = \lim_{b \to \infty} F(b) - F(1) = \lim_{b \to \infty} \frac{g(b)}{b + \ln(b)} - \frac{g(1)}{1 + \ln(1)} = \lim_{b \to \infty} \frac{g(b)}{b + \ln(b)} - 11.
\]

As \( \lim_{b \to \infty} g(b) = \lim_{b \to \infty} (b + \ln(b)) = \infty \), we try to use L’Hôpital’s Rule. We obtain:

\[
\int_1^\infty f(x) \, dx = \lim_{b \to \infty} \frac{g'(b)}{1 + 1/b} - 11 = \frac{21}{1} - 11 = 21 - 11 = 10.
\]

Therefore, the integral converges to 10.