2. [7 points] Let $f(x)$ and $g(x)$ be two continuous and differentiable functions on $[1, \infty)$. Further, suppose these functions have the following properties:

- $F(x) = \dfrac{g(x)}{x + \ln(x)}$ is an antiderivative of $f(x)$ for $x \geq 1$,
- $g(1) = 11$,
- $\lim_{x\to\infty} g(x) = \infty$,
- $\lim_{x\to\infty} g'(x) = 21$.

Compute the value of the following improper integral if it converges. if it does not converge, use a **direct computation** of the integral to show its divergence. Be sure to show your full computation, and be sure to use **proper notation**.

$$\int_1^\infty f(x)\,\mathrm{d}x$$

Circle one: **Diverges** **Converges to** 10

Solution: We start by rewriting this improper integral as a limit, and then use the First Fundamental Theorem of Calculus:

$$\begin{aligned}
\int_1^\infty f(x)\,\mathrm{d}x &= \lim_{b\to\infty} \int_1^b f(x)\,\mathrm{d}x \\
&= \lim_{b\to\infty} F(b) - F(1) \\
&= \lim_{b\to\infty} \frac{g(b)}{b + \ln(b)} - \frac{g(1)}{1 + \ln(1)} \\
&= \lim_{b\to\infty} \frac{g(b)}{b + \ln(b)} - 11.
\end{aligned}$$

As $\lim_{b\to\infty} g(b) = \lim_{b\to\infty} (b + \ln(b)) = \infty$, we try to use L'Hôpital's Rule. We obtain:

$$\begin{aligned}
\int_1^\infty f(x)\,\mathrm{d}x &= \lim_{b\to\infty} \frac{g'(b)}{1 + 1/b} - 11 \\
&= \frac{21}{1} - 11 = 21 - 11 = 10.
\end{aligned}$$

Therefore, the integral converges to 10.