7. [18 points] Determine if the following series converge absolutely, converge conditionally, or diverge. Fully justify your answer including using proper notation and showing mechanics of any tests you use.

a. [8 points] $\sum_{n=1}^{\infty} \frac{\sin(4n)}{4^n}$

Circle one: **Converges Absolutely** **Converges Conditionally** **Diverges**

Solution:
Note first that $\frac{\sin(4n)}{4^n}$ is not always positive, so we should not try to apply the Comparison Test directly. Instead, consider the series $\sum_{n=1}^{\infty} \left|\frac{\sin(4n)}{4^n}\right|$.

For $n \geq 1$, we have $\left|\frac{\sin(4n)}{4^n}\right| \leq \frac{1}{4^n}$, and $\sum_{n=1}^{\infty} \frac{1}{4^n}$ converges as it is a geometric series with common ratio $\frac{1}{4}$.

Therefore, by the (Direct) Comparison Test, $\sum_{n=1}^{\infty} \left|\frac{\sin(4n)}{4^n}\right|$ converges. This means that the original series $\sum_{n=1}^{\infty} \frac{\sin(4n)}{4^n}$ converges absolutely.

7. (continued) Here is a reproduction of the instructions for this problem:
Determine if the following series converge absolutely, converge conditionally, or diverge. Fully justify your answer including using proper notation and showing mechanics of any tests you use.

b. [10 points] $\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^n}{n\ln(n)}$

Circle one: **Converges Absolutely** **Converges Conditionally** **Diverges**

Solution: Note that the series is alternating. Let $a_n = \dfrac{1}{n\ln(n)}$. Then for all n, $0 < a_{n+1} < a_n$, and we also have $\lim\limits_{n\to\infty} a_n = 0$. Therefore, by the alternating series test, $\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^n}{n\ln(n)}$ converges.

Now consider the series $\displaystyle\sum_{n=3}^{\infty} \left|\frac{(-1)^n}{n\ln(n)}\right| = \sum_{n=3}^{\infty} \frac{1}{n\ln(n)}$.

Let $f(x) = \dfrac{1}{x\ln(x)}$. Then $f(x)$ is positive and decreasing. We have:

$$\begin{aligned}
\int_3^{\infty} \frac{1}{x\ln(x)}\,\mathrm{d}x &= \lim_{b\to\infty} \int_3^{b} \frac{1}{x\ln(x)}\,\mathrm{d}x \\
&= \lim_{b\to\infty} \int_3^{b} \frac{1}{x\ln(x)}\,\mathrm{d}x \\
&= \lim_{b\to\infty} \int_{\ln(3)}^{\ln(b)} \frac{1}{u}\,\mathrm{d}u \\
&= \lim_{b\to\infty} \ln(u)\Big|_{\ln(3)}^{\ln(b)} \\
&= \lim_{b\to\infty} \left(\ln(\ln(b)) - \ln(\ln(3))\right) = \infty.
\end{aligned}$$

Therefore, $\displaystyle\int_3^{\infty} \frac{1}{x\ln(x)}\,\mathrm{d}x$ diverges, and so by the Integral test, $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n\ln(n)}$ diverges too.

Therefore $\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^n}{n\ln(n)}$ converges conditionally.