8. [12 points]

a. [7 points] Determine the **radius** of convergence for the following power series. Show all of your work. You do not need to find the interval of convergence.

$$\sum_{n=1}^{\infty}(-1)^n \frac{(2n)!}{9^n (n!)^2} x^{3n}$$

Solution: We use the ratio test:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\frac{9^n}{9^{n+1}}\cdot\frac{(2n+2)!}{(2n)!}\cdot\frac{(n!)^2}{((n+1)!)^2}\cdot\left|\frac{x^{3n+3}}{x^{3n}}\right| \\ &= \lim_{n\to\infty}\frac{1}{9}\cdot\frac{(2n+2)(2n+1)}{(n+1)^2}\cdot\left|x^3\right| \\ &= \frac{4}{9}\left|x^3\right|.\end{aligned}$$

The ratio test tells us the power series converges when this value is smaller than 1, i.e. $\frac{4}{9}\left|x^3\right| < 1$. Rearranging, we see that this implies $|x| < \left(\frac{9}{4}\right)^{1/3}$, which tells us that the radius of convergence is $\left(\frac{9}{4}\right)^{1/3}$.

Answer: $\left(\frac{9}{4}\right)^{1/3}$

b. [5 points] No justification is needed for the remainder of this problem. Suppose that the following is true about the sequence C_n which is defined for $n \geq 0$:

- C_n is monotone decreasing and converges to 0.
- $\displaystyle\sum_{n=0}^{\infty} C_n$ diverges.
- The power series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n C_n}{6^n}(x-5)^n$ has radius of convergence 6.

What is the center of the interval of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n C_n}{6^n}(x-5)^n$?

Answer: 5

What are the endpoints of the interval of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n C_n}{6^n}(x-5)^n$?

Answer: Left endpoint at $c =$ -1

Right endpoint at $d =$ 11

Let c and d be the left and right endpoints of the interval of convergence you found above. Which of the following could be the interval of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n C_n}{6^n}(x-5)^n$? Circle **all** correct answers.

(c, d) [✓ $(c, d]$] $[c, d)$ $[c, d]$