9. [12 points] Katydyd is on vacation from her strenuous bakery job, and is at the beach. She is building a tower out of sand, but periodically sand falls off the top of the tower. Each time sand falls off the tower it gets 25% shorter, and between times sand falls off the top of the tower Katydyd increases its height by 2 inches.

a. [5 points] Let M_n denote the height of Katydyd's tower, in inches, immediately *before* the n^{th} time sand falls off the top of it. Before the first time sand falls off the tower it has a height of 6 inches (so $M_1 = 6$). Find expressions for the values of M_2, M_3 and M_4. You do not need to simplify your expressions.

Solution:

$$M_1 = 6$$
$$M_2 = 0.75M_1 + 2 = 0.75(6) + 2$$
$$M_3 = 0.75M_2 + 2 = 0.75^2(6) + 0.75(2) + 2$$
$$M_4 = 0.75M_4 + 3 = 0.75^3(6) + 0.75^2(2) + 0.75(2) + 2$$

Answer: $M_2 =$ $0.75(6) + 2$

Answer: $M_3 =$ $0.75^2(6) + 0.75(2) + 2$

Answer: $M_4 =$ $0.75^3(6) + 0.75^2(2) + 0.75(2) + 2$

b. [5 points] Find a closed-form expression for M_n. Closed form means your answer should not include ellipses or sigma notation, and should NOT be recursive. You do not need to simplify your expression.

Solution:

$$M_n = 0.75^{n-1}(6) + 2(0.75^{n-2} + \cdots + 0.75 + 1)$$
$$= 0.75^{n-1}(6) + \frac{2(1 - 0.75^{n-1})}{1 - 0.75}$$

Answer: $M_n =$ $0.75^{n-1}(6) + \frac{2(1 - 0.75^{n-1})}{1 - 0.75}$

c. [2 points] If Katydyd were to keep doing this indefinitely, what height would her tower approach, in inches, in the long run?

Solution:

$$\lim_{n\to\infty} M_n = \lim_{n\to\infty} 0.75^{n-1}(6) + \frac{2(1 - 0.75^{n-1})}{1 - 0.75} = \frac{2}{0.25} = 8$$

Answer: **8**