3. [8 points]

a. [5 points] Show that the following limit converges and compute the limit. Fully justify your answer including using **proper limit notation**.

$$\lim_{n\to\infty} n^2\left(1-\cos\left(\frac{1}{n}\right)\right)$$

Solution:
We use L'Hospital's Rule twice to find this limit.

$$\begin{aligned}
\lim_{n\to\infty} n^2\left(1-\cos\left(\frac{1}{n}\right)\right) &= \lim_{n\to\infty} \frac{1-\cos\left(\frac{1}{n}\right)}{\frac{1}{n^2}} \\
&\overset{\text{LH}}{=} \lim_{n\to\infty} \frac{\sin(\frac{1}{n})(-\frac{1}{n^2})}{-\frac{2}{n^3}} \\
&= \frac{1}{2}\lim_{n\to\infty} n\sin\left(\frac{1}{n}\right) \\
&= \frac{1}{2}\lim_{n\to\infty} \frac{\sin(\frac{1}{n})}{\frac{1}{n}} \\
&\overset{\text{LH}}{=} \frac{1}{2}\lim_{n\to\infty} \cos\left(\frac{1}{n}\right) \\
&= \frac{1}{2}
\end{aligned}$$

Answer: $\frac{1}{2}$

b. [3 points] Use part **a.** to determine if the following series converges or diverges, and circle the corresponding word. **Fully justify** your answer including using **proper notation** and showing mechanics of any tests you use.

$$\sum_{n=1}^{\infty} n^2\left(1-\cos\left(\frac{1}{n}\right)\right)$$

Circle one: **Converges** **Diverges**

Justification:

Solution: From **a.** we know that

$$\lim_{n\to\infty} n^2\left(1-\cos\left(\frac{1}{n}\right)\right) = \frac{1}{2} \neq 0$$

and so by the nth term test for divergence, $\sum_{n=1}^{\infty} n^2\left(1-\cos\left(\frac{1}{n}\right)\right)$ diverges.