

8. [12 points] Melissa runs an operation to remove pesticides from a local lake. Initially there are 200kg of pesticides in the lake. The operation runs from 10am till 10pm each day, and by 10pm each day, the mass of pesticides in the lake is always $p\%$ lower than it was at 10am, where p is a constant. From 10pm till 10am each night, runoff from local rivers adds 12kg of pesticides back into the lake.
- a. [5 points] Let S_n denote the total mass of pesticides, in kg, in the lake at 10pm on the n th day of operation. Find expressions for the values of S_1 , S_2 and S_3 . You do not need to simplify your expressions, and they may be given in terms of p .

Solution: We see that $S_1 = 200 \left(1 - \frac{p}{100}\right) = 200 - 2p$.

Therefore $S_2 = (S_1 + 12) \left(1 - \frac{p}{100}\right) = 200 \left(1 - \frac{p}{100}\right)^2 + 12 \left(1 - \frac{p}{100}\right)$, and so $S_3 = (S_2 + 12) \left(1 - \frac{p}{100}\right) = 200 \left(1 - \frac{p}{100}\right)^3 + 12 \left(1 - \frac{p}{100}\right)^2 + 12 \left(1 - \frac{p}{100}\right)$.

Answer: $S_1 = \underline{200 \left(1 - \frac{p}{100}\right)}$

Answer: $S_2 = \underline{200 \left(1 - \frac{p}{100}\right)^2 + 12 \left(1 - \frac{p}{100}\right)}$

Answer: $S_3 = \underline{200 \left(1 - \frac{p}{100}\right)^3 + 12 \left(1 - \frac{p}{100}\right)^2 + 12 \left(1 - \frac{p}{100}\right)}$

- b. [5 points] Find a closed-form expression for S_n . Closed form means your answer should not include ellipses or sigma notation, and should NOT be recursive. You do not need to simplify your expression and it may be given in terms of p .

Solution: Following our intuition from part a., we can see that

$$S_n = 200 \left(1 - \frac{p}{100}\right)^n + 12 \left(1 - \frac{p}{100}\right)^{n-1} + \cdots + 12 \left(1 - \frac{p}{100}\right).$$

Using the formula for the sum of a finite geometric series,

$$\begin{aligned} S_n &= 200 \left(1 - \frac{p}{100}\right)^n + \frac{12 \left(1 - \frac{p}{100}\right) \left(1 - \left(1 - \frac{p}{100}\right)^{n-1}\right)}{1 - \left(1 - \frac{p}{100}\right)} \\ &= 200 \left(1 - \frac{p}{100}\right)^n + \frac{1200}{p} \left(1 - \frac{p}{100}\right) \left(1 - \left(1 - \frac{p}{100}\right)^{n-1}\right) \end{aligned}$$

Answer: $S_n = \underline{200 \left(1 - \frac{p}{100}\right)^n + \frac{1200}{p} \left(1 - \frac{p}{100}\right) \left(1 - \left(1 - \frac{p}{100}\right)^{n-1}\right)}$

- c. [2 points] Melissa aims to make sure that, in the long run, the mass of pesticides in the lake at 10pm each day approaches 28kg. What is the smallest value of p that will ensure that Melissa meets her goal? Show all of your work.

Solution: We note that $\lim_{n \rightarrow \infty} S_n = \frac{1200}{p} \left(1 - \frac{p}{100}\right) = \frac{1200}{p} - 12$. We want $\frac{1200}{p} - 12 = 28$, and thus $p = 30$.

Answer: $p = \underline{30}$