9. [10 points]

a. [7 points] Determine the **radius** of convergence for the following power series. Show all of your work. You do not need to find the interval of convergence.

$$\sum_{n=1}^{\infty}(-1)^n \frac{(3n)!}{5^n((n+1)!)^3}(x-4)^{2n}$$

Solution: We denote

$$a_n = (-1)^n \frac{(3n)!}{5^n((n+1)!)^3}(x-4)^{2n}.$$

We see that:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\frac{(3n+3)!}{5^{n+1}((n+2)!)^3}\frac{5^n((n+1)!)^3}{(3n)!}\frac{|x-4|^{2n+2}}{|x-4|^{2n}}\\ &= \lim_{n\to\infty}\frac{5^n}{5^{n+1}}\frac{(3n+3)!}{(3n)!}\frac{((n+1)!)^3}{((n+2)!)^3}|x-4|^2\\ &= \lim_{n\to\infty}\frac{1}{5}\frac{(3n+3)(3n+2)(3n+1)}{(n+2)^3}|x-4|^2\\ &= \frac{27}{5}|x-4|^2,\end{aligned}$$

and so by the Ratio test, the power series converges for $\frac{27}{5}|x-4|^2 < 1$, i.e. for $|x-4| < \left(\frac{5}{27}\right)^{1/2}$.

Similarly, the power series diverges for $|x-4| > \left(\frac{5}{27}\right)^{1/2}$.

Therefore, the radius of convergence is $\left(\frac{5}{27}\right)^{1/2}$.

Answer: $\left(\frac{5}{27}\right)^{1/2}$

b. [3 points] Suppose the power series $\sum_{n=0}^{\infty} C_n(x-a)^n$ has radius of convergence 3, and that the series diverges for $x=7$ and converges for $x=10$. Which of the following could be the value of a? Circle **all** correct options.

i. 4 ii. 7 iii. [10] iv. [13] v. 16 vi. NONE OF THESE