6. (10 pts) Does $\frac{1}{3\ln(3)} + \frac{1}{4\ln(4)} + \frac{1}{5\ln(5)} + \frac{1}{6\ln(6)} + \ldots$ converge or diverge? Demonstrate unequivocally that your answer is correct.

Integral test:

$$\sum_{n=3}^{\infty} \frac{1}{n\ln(n)} \text{ converges if and only if } \int_3^{\infty} \frac{dx}{x\ln(x)} \text{ converges.}$$

So let's look at that integral. If we substitute $w = \ln(x)$, then $dw = dx/x$, so

$$\int \frac{dx}{x\ln(x)} = \int \frac{dw}{w} = \ln(w) + C = \ln(\ln(x)) + C.$$

That means

$$\int_3^{\infty} \frac{dx}{x\ln(x)} = \lim_{b\to\infty} \ln(\ln(x))\Big|_3^b = \lim_{b\to\infty} \ln(\ln(b)) - \ln(\ln(3)).$$

Now for large x, $\ln(x)$ is proportional to the number of digits in x. So it does go to infinity as x gets large, but very slowly. That means that $\ln(\ln(x))$ is like the number of digits in the number of digits in x. So it, too, goes to infinity, but really, *really* slowly. Nevertheless, the integral diverges, so the sum diverges as well.