10. (10 points) The *electric potential* is a quantity of great importance in electrostatics. The electric potential $V(R)$ at a distance $R$ along the axis perpendicular to the center of a charged disk with radius 1 is given by

$$V(R) = C \left( \sqrt{R^2 + 1} - R \right)$$

where $C$ is a constant that depends on the choice of units that are being used.

(a) Show that for large numbers $R$,

$$V(R) \approx \frac{C}{2R}.$$  

(Hint: $\sqrt{R^2 + 1} = R\sqrt{1 + \frac{1}{R^2}}$ and remember that $R$ is large.)

As the hint says, $\sqrt{R^2 + 1} = R\sqrt{1 + \frac{1}{R^2}} = R(1 + \frac{1}{R^2})^{\frac{1}{2}}$.

Recall (or calculate) that for $-1 < x < 1$, we have $(1 + x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \ldots$.

Thus for $R$ large (enough so that $-1 < \frac{1}{R^2} < 1$) we have

$$(1 + \frac{1}{R^2})^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \ldots = 1 + \frac{1}{2}R - \frac{1}{8}R^2 + \frac{1}{16}R^3 - \ldots$$

So $R(1 + \frac{1}{R^2})^{\frac{1}{2}} = R + \frac{1}{2}R - \frac{1}{8}R^2 + \frac{1}{16}R^3 - \ldots$

Therefore $R(1 + \frac{1}{R^2})^{\frac{1}{2}} - R = \frac{1}{2}R - \frac{1}{8}R^2 + \frac{1}{16}R^3 - \ldots$

So $C(R(1 + \frac{1}{R^2})^{\frac{1}{2}} - R) = \frac{C}{2}R - \frac{C}{8}R^2 + \frac{C}{16}R^3 - \ldots$

And so we have that for large numbers $R$, we can approximate $C(R(1 + \frac{1}{R^2})^{\frac{1}{2}} - R)$ by $\frac{C}{2R}$.

(b) Approximately how large should $R$ be in order that the error in the approximation of $V(R)$ by $C/2R$ is less than 1% of $V(R)$?

For large $R$, the error in the approximation of $V(R)$ by $C/2R$ is approximately $-\frac{C}{8R^3}$.

So, we want to approximately solve

$$V(R) = \frac{C}{100}$$

for $R$.

This is approximately the same as approximately solving

$$\frac{C}{2R} = \frac{C}{8R^3}$$

for $R$.

So we want to solve

$$4R^2 = 100$$

and thus

R should approximately be greater than or equal to 5.