4. (8 points) Show that if \( \sum_{n=1}^{\infty} a_n \) converges, then \( \lim_{n \to \infty} a_n = 0. \)

Let \( S_n = a_1 + a_2 + \cdots + a_n \), so that \( S_n - S_{n-1} = a_n. \) The series converges to \( S \) if and only if \( S = \lim_{n \to \infty} S_n \) exists. In this case, \( \lim_{n \to \infty} S_n - S_{n-1} = S \) as well so

\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} S_n - S_{n-1} = \lim_{n \to \infty} S_n - \lim_{n \to \infty} S_{n-1} = S - S = 0.
\]

5. (8 points) In this question we will investigate the convergence of the power series \( \sum_{n=0}^{\infty} \frac{n^2}{e^n} (x + 2)^n. \)

(a) Find the radius of convergence, \( R \), of the power series. (Show your work.)

**Solution.** We will use the ratio test. Let \( a_n = \frac{n^2(x + 2)^n}{e^n} \) so that

\[
\frac{a_{n+1}}{a_n} = \frac{(n+1)^2(x+2)^{n+1}}{e^{n+1}} \cdot \frac{e^n}{n^2(x+2)^n} = \frac{(n+1)^2(x+2)}{e^2}.
\]

Consequently,

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{(n+1)^2}{e^n 2^n} |x+2| = \frac{|x+2|}{e}.
\]

By the ratio test, the series therefore converges when this limit is less than one and diverges when it is greater than one. That is, the series converges for \( |x+2| < e \) and diverges for \( |x+2| > e \), so the radius of convergence is \( R = e \). The series converges for \(-e < x + 2 < e\) or \(-2 - e < x < -2 + e\).

\[
R = \boxed{e}.
\]

(b) What is the interval of convergence of the power series?

\[-2 - e < x < -2 + e.\]