4. (8 points) Show that if $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Let $S_n = a_1 + a_2 + \cdots + a_n$, so that $S_n - S_{n-1} = a_n$. The series converges to S if and only if $S = \lim_{n\to\infty} S_n$ exists. In this case, $\lim_{n\to\infty} S_{n-1} = S$ as well so

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} S_n - S_{n-1} = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0.$$

5. (8 points) In this question we will investigate the convergence of the power series $\sum_{n=0}^{\infty} \frac{n^2}{e^n}(x+2)^n$.

(a) Find the radius of convergence, R, of the power series. (Show your work.)

Solution. We will use the ratio test. Let $a_n = \frac{n^2(x+2)^n}{e^n}$ so that

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)^2(x+2)^{(n+1)}}{e^n}}{\frac{n^2(x+2)^n}{e^n}} = \frac{(n+1)^2(x+2)}{en^2}.$$

Consequently,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)^2}{en^2}|x+2| = \frac{|x+2|}{e}.$$

By the ratio test, the series therefore converges when this limit is less than one and diverges when it is greater than one. That is, the series converges for $|x+2| < e$ and diverges for $|x+2| > e$, so the radius of convergence is $R = e$. The series converges for $-e < x + 2 < e$ or $-2 - e < x < -2 + e$.

$R =$ $\boxed{\text{e}}$.

(b) What is the interval of convergence of the power series?

$$\boxed{-2 - e < \quad x \quad < -2 + e.}$$