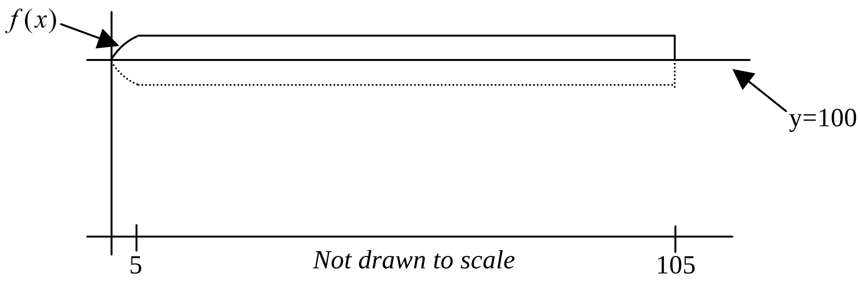
2. (60 points) Several different forces act on a submerged submarine and cause it to rise and/or fall. In this problem, we will use a simplified model of a submarine to explore some of these forces. To construct our model, we revolve the graph of

$$f(x)=\begin{cases} 100+\sqrt{5}e^{0.5}\sqrt{x}e^{-0.1x} & 0\le x\le 5 \\ 105 & 5<x\le 105 \end{cases}$$

around $y=100$ (see picture below). Note that all units in the horizontal and vertical directions are measured in meters. We will also assume that ocean water has density of $1025\,{}^{kg}/_{m^3}$ and that the density of material inside the submarine is a constant represented by the symbol δ. Additionally, the acceleration due to gravity is $9.8\,{}^{m}/_{sec^2}$.



a. The *force due to buoyancy* is an upward force equal to the weight of the water displaced by the volume of the submarine. Find the volume of water displaced by the sub (i.e. the volume of the submarine) and the resulting force due to the buoyancy.

On $x\in[0,5]$,

$$V_{slice}=\pi r^2\Delta x=\pi\left(f(x)-100\right)^2\Delta x=\pi 5e^{1.0}xe^{-0.2x}\Delta x.$$

So total volume is $\lim\limits_{\Delta x\to 0}\sum \pi 5e^1 xe^{-0.2x}\Delta x=5\pi e^1\int_0^5 xe^{-0.2x}\,dx$. Use integration by parts with $u=x$ and $v'=e^{-0.2x}$ to get

$$5\pi e\int_0^5 xe^{-0.2x}dx=5\pi e\left[\left. x\frac{e^{-0.2x}}{-0.2}\right|_0^5-\int_0^5\frac{e^{-0.2x}}{-0.2}dx\right]$$

$$=5\pi e\left[x\frac{e^{-0.2x}}{-0.2}-\frac{e^{-0.2x}}{0.04}\right]_0^5=5\pi e\left[5\frac{e^{-1}}{-0.2}-\frac{e^{-1}}{0.04}-0+\frac{e^0}{0.04}\right]$$

$$=-250\pi+125\pi e=282.0686m^3$$

On $x\in(5,105]$, we've got a cylinder with radius 5. So volume is $2500\pi m^3$

Total volume is $2250\pi+125\pi e=8136.05m^3$

Convert this to a force by multiplying by density and acceleration due to gravity to get

$1025\cdot 9.8\cdot\left(2250\pi+125\pi e\right)=81,726,624.745N$

2. (b) Find the center of mass of the nose of the submarine (i.e. the shape of the first 5 meters of our model). Note *it is only necessary to set up, but not calculate*, (an) integral(s).

$$\bar{x} = \frac{\int_0^5 x\delta A_x(x)\,dx}{\text{mass}} = \frac{\int_0^5 x\delta\pi\left(\sqrt{5}e^{0.5}\sqrt{x}e^{-0.1x}\right)^2\,dx}{\text{mass}} = \frac{5e\pi\delta\int_0^5 x^2e^{-0.2x}\,dx}{\text{mass}}$$

where

$$\text{mass} = 1025(2250\pi + 125\pi e) = 8339451.5\,\text{kg}.$$

By symmetry, $\bar{y} = \bar{z} = 0$. In this case, I'm assuming the z direction is perpendicular to the usual x and y axes.

3. (60 points) The following questions refer to the submarine described in problem #2.

(a) The buoyancy properties of the empty submarine described in problem 2 cause the submarine to begin moving upward through the ocean water. This motion, in conjunction with the ocean water, creates a damping force that begins to slow the submarine. Assume that the damping force is proportional to the square of the velocity of the submarine, and that when the velocity is 5 m/s the force is 100 N. For our model submarine, the velocity at t seconds can be described by

$$v(t) = \left(25 - 25\sin\left(\frac{\pi t}{60}\right)\right)^{\frac{1}{3}} \quad \text{meters per second.}$$

Find the amount of work the damping force does on the submarine over the first 30 seconds of motion.

From the problem statement, the damping force is kv^2 where k is the proportionality constant. Since the force is 100 N when the velocity is 5 m/s, we solve and find that

$$k = \frac{100\,\text{N}}{(5\,\text{m/s})^2} = 4\,\text{kg/m}.$$

The distance travelled from time t to time $t + \Delta t$ is approximately $v(t)\Delta t$. Thus the work done over that slice of time is

$$\text{Force}\cdot\text{Distance} = (kv^2)(v\Delta t) = kv^3\Delta t = 4\left(25 - 25\sin\left(\frac{\pi t}{60}\right)\right)\Delta t$$

which means the total work is

$$\begin{aligned}\int_0^{30} 4\left(25 - 25\sin\left(\frac{\pi t}{60}\right)\right)dt = 100\int_0^{30}\left(1 - \sin\left(\frac{\pi t}{60}\right)\right)dt &= 100\left[t + \frac{60}{\pi}\cos\left(\frac{\pi t}{60}\right)\right]_0^{30} \\ &= 100\left[\left(30 + \frac{60}{\pi}\cos\frac{\pi}{2}\right) - \left(0 + \frac{60}{\pi}\cos 0\right)\right] \\ &= 100(30 - 60/\pi) \approx \boxed{1090.14\,\text{Joules}}.\end{aligned}$$