4. (50 points) The *Erlang k-distribution* is a probability distribution often used in mathematical modeling when events happen at a roughly (but not exactly) constant rate. It is a good model for the wait times at a telephone switchboard when calls come in on average every λ seconds. In this case, the wait time for the next *k* telephone calls has a probability density function that is the Erlang k-distribution

$$f_{k,\lambda}(x) = \begin{cases} \dfrac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!} & x \geq 0 \\ 0 & x < 0 \end{cases}.$$

a. For $x \geq 0$, the Erlang k- distribution has the non-obvious cumulative distribution function,

$$C_{k,\lambda}(x) = 1 - \sum_{n=0}^{k-1} \frac{e^{-\lambda x}(\lambda x)^n}{n!}.$$

Using an appropriate test, show that the sum in the cumulative distribution function converges as $k \to \infty$, thus verifying that $C_{k,\lambda}(x)$ is finite. You may assume $x = 1$ and $\lambda = 3$. Note: saying "the distribution function must be finite, therefore it converges" will not be given credit.

As $k \to \infty$, the sum in the cumulative distribution becomes $\sum_{n=0}^{\infty} \frac{e^{-3} 3^n}{n!}$

Using the ratio test, we get $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{e^{-3} 3^{n+1}}{(n+1)!} \cdot \frac{n!}{e^{-3} 3^n} \right| = \lim_{n\to\infty} \left| \frac{3}{n+1} \cdot \frac{1}{1} \right| = 0$ since the numerator is constant and the denominator grows to infinity.

Since $L = 0 < 1$, the ratio test allows us to conclude that the original sum *converges*

b. A call arrives at the switchboard at 2:38:06pm. Assuming $k=1$ and $\lambda=3$ seconds, find the probability that the next phone call comes in between 2:38:08pm and 2:38:09pm.

Using the c.d.f, we simply subtract

$$C_{1,3}(3)-C_{1,3}(2)=-\frac{e^{-3(3)}(3\cdot 3)^0}{0!}+\frac{e^{-3(2)}(3\cdot 2)^0}{0!}=\frac{-1}{e^9}+\frac{1}{e^6}=.002355$$ to get 0.2355%

Using the p.d.f, we compute

$$\int_2^3 f_{1,3}(x)dx=\int_2^3 \frac{3^1x^0e^{-3x}}{0!}dx=3\int_2^3 e^{-3x}dx=3\left[\frac{e^{-3x}}{-3}\right]_2^3=-e^{-9}+e^{-6}=0.002355$$ which is 0.2355%