5. (52 points) Rigorously determine whether or not the following converge or diverge. You should make clear to the grader any thoughts and/or processes that you use.

a. $\sum_{n=6}^{\infty} \frac{\ln(n)+3}{n-4}$

Note that $\frac{\ln(n)+3}{n-4} > \frac{\ln(n)}{n-4} > \frac{1}{n-4} > \frac{1}{n}$. This implies that $\sum_{n=6}^{\infty} \frac{\ln(n)+3}{n-4} > \sum_{n=6}^{\infty} \frac{1}{n}$. We know by the *p*-test that $\sum_{n=6}^{\infty} \frac{1}{n}$ diverges (p = 1). Thus by the comparison test, since $\sum_{n=6}^{\infty} \frac{\ln(n)+3}{n-4}$ is larger than $\sum_{n=6}^{\infty} \frac{1}{n}$, $\sum_{n=6}^{\infty} \frac{\ln(n)+3}{n-4}$ must diverge as well.

b. $\sum_{n=1}^{\infty} \frac{n+\sin(n)+1}{e^n-n-1}$

We use the limit comparison test with $b_n = \frac{n}{e^n}$. Now

$\lim_{n\to\infty} \frac{\frac{n+\sin(n)+1}{e^n-n-1}}{\frac{n}{e^n}} = \lim_{n\to\infty} \frac{e^n}{e^n-n-1} \cdot \frac{(n+\sin(n)+1)}{n}$ But $\frac{e^n}{e^n-n-1}$ approaches 1 as n gets large since exponential functions dominate polynomial functions. And since $-1 \le \sin(n) \le 1$, $\frac{(n+\sin(n)+1)}{n}$ approaches 1 as n grows large. Thus

$\lim_{n\to\infty} \frac{e^n}{e^n-n-1} \cdot \frac{(n+\sin(n)+1)}{n} = 1\cdot 1 = 1$. Since this limit is finite and non-zero, we know that $\sum_{n=1}^{\infty} \frac{n+\sin(n)+1}{e^n-n-1}$ and $\sum_{n=1}^{\infty} \frac{n}{e^n}$ both converge or both diverge. Apply ratio test .

$\lim_{n\to\infty} \left| \frac{n+1}{e^{n+1}} \cdot \frac{e^n}{n} \right| = \lim_{n\to\infty} \left| \frac{n+1}{n} \cdot \frac{1}{e} \right| = \frac{1}{e} < 1$ .Since the limit is less than 1, $\sum_{n=1}^{\infty} \frac{n}{e^n}$ converges by the ratio test and thus by the limit comparison test $\sum_{n=1}^{\infty} \frac{n+\sin(n)+1}{e^n-n-1}$ converges.