5. [10 points] Let t be the number of minutes a student waits for the Bursley-Baits bus. For constants a and b, the probability density function giving the distribution of t is

$$p(t) = \begin{cases} 0 & \text{if } t < 0 \\ ae^{-bt} & \text{if } 0 \le t < \infty. \end{cases}$$

According to this density function, the mean waiting time for the bus is 8 minutes.

a. [6 points] Determine the exact values of the constants a and b. Answers supported only by calculator work will not receive full credit. Write your final answers on the spaces provided.

Solution: Because this is a density function, $\int_0^\infty ae^{-bt}dt = 1$. We have

$$\lim_{c\to\infty} \int_0^c ae^{-bt}dt = \lim_{c\to\infty} -\frac{a}{b}e^{-bt}|_0^c = \lim_{c\to\infty} \left[\frac{a}{b} - \frac{a}{b}e^{-bc}\right] = \frac{a}{b} = 1.$$

This gives us the condition $a = b$.
We know the mean time is 8, so we also have $8 = \int_0^\infty tae^{-bt}dt$. Then we have $8 = \lim_{c\to\infty} \int_0^c ate^{-bt}dt$. Use integration by parts using $u = at$, $du = adt$, $dv = e^{-bt}dt$, $v = -\frac{1}{b}e^{-bt}$. This gives us

$$\lim_{c\to\infty} \left(-\frac{a}{b}te^{-bt}|_0^c + \int_0^c \frac{a}{b}e^{-bt}dt\right) = \lim_{c\to\infty} \left(-\frac{a}{b}ce^{-bc} - \frac{a}{b^2}e^{-bt}|_0^c\right)$$

$$= \lim_{c\to\infty} \left(-\frac{a}{b}ce^{-bc} - \frac{a}{b^2}e^{-bc} + \frac{a}{b^2}\right) = \frac{a}{b^2} = 8.$$

Using $a = b$, we have $8 = \frac{b}{b^2} = \frac{1}{b}$, so that $b = \frac{1}{8} = a$.

$a =$ **1/8** $\qquad$ $b =$ **1/8**

b. [4 points] Using your answers from part (a), determine the exact value for median waiting time. Include units in your answer. Answers supported only by calculator work will not receive full credit.

Solution: Let M be the median so that

$$0.5 = \int_0^M \frac{1}{8}e^{-t/8}dt.$$

Integrating we get

$$0.5 = -e^{-t/8}|_0^M = -e^{-M/8} + 1.$$

This gives us $e^{-M/8} = 0.5$, so that $-\frac{M}{8} = \ln(0.5)$, and $M = -8\ln(0.5) = 8\ln(2)$ minutes.