6. [12 points] The position of a particle at time t is given by $x = \cos\left(e^t\right)$, and $y = \cos\left(3e^t\right)$, where both x and y are measured in cm, and t is measured in seconds.

a. [5 points] Find the exact speed of the particle at time $t = 0$. Show enough work to support your answer and include units. Calculator approximations will not receive full credit.

Solution: We first find $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$$\frac{dx}{dt} = -e^t \sin(e^t) \text{ and } \frac{dy}{dt} = -3e^t \sin(3e^t)$$

$$\text{Speed } = \sqrt{(-e^t \sin(e^t))^2 + (-3e^t \sin(3e^t))^2} = \sqrt{e^{2t}\sin^2(e^t) + 9e^{2t}\sin^2(3e^t)}$$

Evaluating at $t = 0$ gives us Speed $= \sqrt{\sin^2(1) + 9\sin^2(3)}$ cm/sec.

b. [7 points] Use derivatives to determine the concavity of the particle's path at time $t = 0$.

Solution:

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) / \frac{dx}{dt}.$$

From part (a), $\frac{dy}{dx} = \frac{-3e^t \sin(3e^t)}{-e^t \sin(e^t)} = \frac{3\sin(3e^t)}{\sin(e^t)}$.

$$\frac{d}{dt}\left(\frac{3\sin(3e^t)}{\sin(e^t)}\right) / \frac{dx}{dt} = \frac{9e^t \cos(3e^t)\sin(e^t) - 3e^t \sin(3e^t)\cos(e^t)}{\sin^2(e^t)} / (-e^t \sin(e^t))$$

$$= \frac{9e^t \cos(3e^t)\sin(e^t) - 3e^t \sin(3e^t)\cos(e^t)}{-e^t \sin^3(e^t)}.$$

Evaluating at $t = 0$, we have $\frac{d^2y}{dx^2} = \frac{9\cos(3)\sin(1) - 3\sin(3)\cos(1)}{-\sin^3(1)} \approx 12.9673 > 0$. Therefore the path is concave up when $t = 0$.