9. [14 points] An ice cube melts at a rate proportional to its surface area. Let $V(t)$ denote the volume (in cm^3) of the ice cube, and let $x(t)$ denote the length (in cm) of a side of the ice cube t seconds after it begins to melt.

a. [4 points] Write a differential equation for $V(t)$, the ice cube's volume t seconds after it started melting. Your differential equation may contain V, t and an unknown constant k.

Solution: We know that $V = x^3$, so $x = V^{1/3}$. The surface area of the cube is given by $6x^2$. That gives us $\frac{dV}{dt} = 6kx^2$, and substituting x in terms of V, we have $\frac{dV}{dt} = 6kV^{2/3}$.

b. [4 points] The ice cube's initial volume is $V_0 > 0$. Solve the differential equation you found in part (a), finding V in terms of t, k, and V_0.

Solution: Using separation of variables, we have $\frac{dV}{V^{2/3}} = 6kdt$. This gives $3V^{1/3} = 6kt + C$, and $V^{1/3} = 2kt + C$, or $V = (2kt + C)^3$. When $t = 0$, $V = V_0$, which gives us $V_0 = C^3$, so that $C = V_0^{1/3}$. The solution is then $V = (2kt + V_0^{1/3})^3$.

c. [6 points] Graph the volume of the ice cube versus time given $V(0) = V_0$. Be sure to label your axes and any important features of your graph, including the time at which the ice cube has completely melted.

Solution: The vertical intercept is $V = V_0$. The horizontal intercept is $t = -\frac{1}{2k}V_0^{1/3}$.