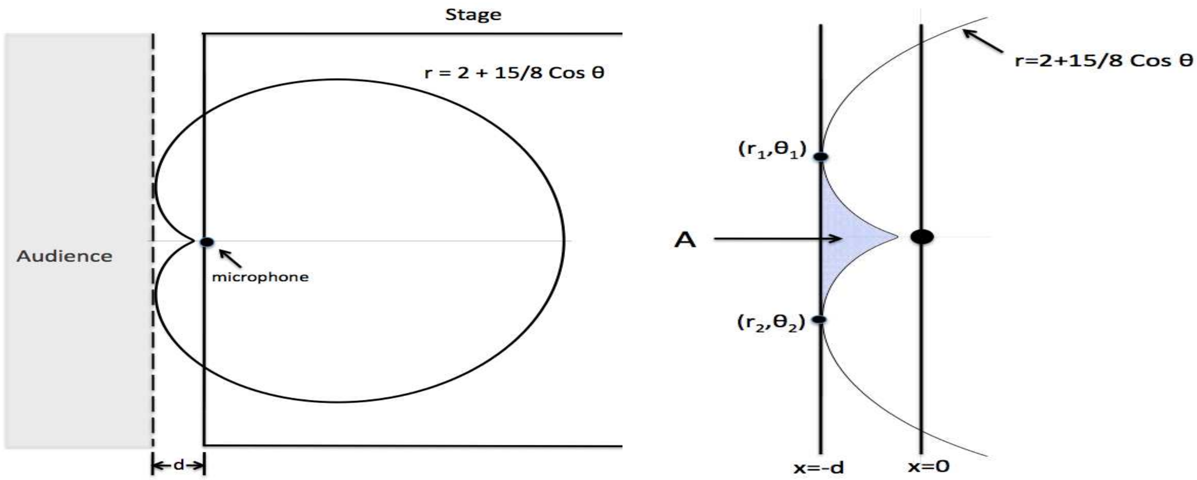
2. [14 points] A microphone at the point $r = 0$ detects sounds in a region enclosed by the cardioid $r = 2 + \frac{15}{8}\cos\theta$. The microphone is placed in front of the stage at an auditorium to record a musical band. Let d denote the smallest distance you must leave between the audience and the microphone to avoid recording any noise from the public in attendance.



a. [5 points] Write an integral that computes the area of the shaded region A in terms of θ_1, θ_2 and d.

Solution: The line $x = -d$ in polar coordinates is $r = \frac{-d}{\cos\theta}$. Hence

$$A = \int_{\theta_1}^{\theta_2} \frac{1}{2}\left(\frac{-d}{\cos\theta}\right)^2 d\theta - \int_{\theta_1}^{\theta_2} \frac{1}{2}\left(2 + \frac{15}{8}\cos\theta\right)^2 d\theta$$

b. [4 points] Write a formula in terms of θ that computes the value of the slope of the tangent line to the cardioid.

Solution:

$$\frac{dy}{dx} = \frac{y'}{x'} = \frac{((2 + \frac{15}{8}\cos\theta)\sin\theta)'}{((2 + \frac{15}{8}\cos\theta)\cos\theta)'} = \frac{(-\frac{15}{8}\sin\theta)\sin\theta + (2 + \frac{15}{8}\cos\theta)\cos\theta}{(-\frac{15}{8}\sin\theta)\cos\theta - (2 + \frac{15}{8}\cos\theta)\sin\theta}$$

c. [3 points] Find an exact expression for the values of $0 \le \theta < 2\pi$ at which the cardioid has a vertical tangent line. Full credit will not be given for decimal approximations.

Solution: $x' = (-\frac{15}{8}\sin\theta)\cos\theta - (2 + \frac{15}{8}\cos\theta)\sin\theta = -\sin\theta(2 + \frac{15}{4}\cos\theta) = 0$.
$\sin\theta = 0$ then $\theta = 0, \pi$.
$2 + \frac{15}{4}\cos\theta = 0$ then $\cos\theta = -\frac{8}{15}$.
This yields $\theta = 0, \pi$, $\theta_1 = \cos^{-1}\left(-\frac{8}{15}\right)$, $\theta_2 = 2\pi - \cos^{-1}\left(-\frac{8}{15}\right)$

d. [2 points] Find the value of d. Show all your work.

Solution: $d = -x(\theta_1) = -(2 + \frac{15}{8}\cos\theta_1)\cos\theta_1 = \frac{8}{15}$