3. [14 points] A farmer notices that a population of grasshoppers is growing at undesirable levels in his crop. He decides to hire the services of a pest control company. They offer the farmer a pesticide capable of eliminating the grasshoppers at a rate of 1 thousand grasshoppers per week. In the absence of pesticides, it is estimated that the grasshopper population grows at a rate of 20 percent every week. Let $P(t)$ be the number of grasshoppers (in thousands) t weeks after the pesticide is applied to the crop. Then $P(t)$ satisfies

$$\frac{dP}{dt} = \frac{P}{5} - 1.$$

Suppose there are P_0 thousand grasshoppers in the crop at the time the pesticide is applied in the crop.

a. [8 points] Find a formula for $P(t)$ in terms of t and P_0.

Solution:

$$\begin{aligned}
\frac{dP}{dt} &= \frac{P}{5} - 1. \\
\frac{dP}{dt} &= \frac{1}{5}(P-5) \\
\frac{dP}{P-5} &= \frac{1}{5}dt \\
\ln|P-5| &= \frac{1}{5}t + C \\
P - 5 &= Be^{\frac{1}{5}t} \quad P(0) = P_0 = 5 + B \quad B = P_0 - 5. \\
P(t) &= 5 + (P_0 - 5)e^{\frac{1}{5}t}.
\end{aligned}$$

b. [3 points] Does the differential equation have any equilibrium solutions? List each equilibrium solution and determine whether it is stable or unstable. **Justify your answer**.

Solution: Equilibrium solutions: $P(t) = 5$. The equilibrium is unstable since for $P_0 > 5$ $P(t)$ increases and for $P_0 < 5$ $P(t)$ decreases.

c. [3 points] Does the effectiveness of the pesticide depend on P_0? That is, is the pesticide guaranteed to eliminate the grasshopper population regardless of the value of P_0, or are there some values of P_0 for which the grasshoppers will survive? If so, determine these values of P_0.

Solution: The pesticide is effective if $P_0 < 5$ and ineffective if $P_0 \geq 5$.