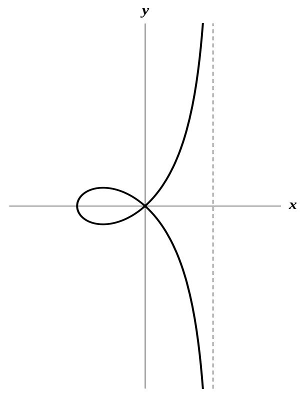
3. [10 points] The motion of a particle is given by the following parametric equations



$$x(t) = \frac{a(t^2-1)}{t^2+1} \qquad\qquad y(t) = \frac{t^3-t}{t^2+1}.$$

for $-\infty < t < \infty$ and a positive constant a. Show all your work to receive full credit.

a. [3 points] Find the values of t at which the particle passes through the origin.

Solution: We need to solve simultaneously $x(t) = 0$ and $y(t) = 0$.

$\bullet x(t) = 0$, then $\frac{a(t^2-1)}{t^2+1} = 0$. This is only possible if $t^2 - 1 = 0$. Hence $t = \pm 1$.

$\bullet y(t) = 0$, then $\frac{t^3-t}{t^2+1} = 0$. This is only possible if $t^3 - t = 0$. Hence $t = 0, \pm 1$.

Times at the origin $t = \pm 1$.

b. [5 points] Find the value of t at which the curve defined by the parametric equations has a vertical tangent line. Also, give the (x, y) coordinates of this point.

Solution:

$$x'(t) = a\left[\frac{(t^2+1)(2t) - (t^2-1)(2t)}{(t^2+1)^2}\right] = \frac{4at}{(t^2+1)^2} \qquad \text{then} \quad x'(t) = 0 \qquad \text{at } t = 0.$$

and $(x(0), y(0)) = (-a, 0)$.

c. [2 points] The curve has a vertical asymptote. Find the equation of this asymptote.

Solution: $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \frac{a(t^2-1)}{t^2+1} = \lim_{t\to\infty} \frac{at^2}{t^2} = a$. Then the equation of the vertical asymptote is $x = a$.