4. [13 points]

a. [6 points] A cylindrical tank with height 8 m and radius of 8 m is standing on one of its circular ends. The tank is initially empty. Water is added at a rate of 2 m^3 /min. A valve at the bottom of the tank releases water at a rate proportional to the water's depth (proportionality constant $= k$). Let $V(t)$ be the volume of the water in the tank at time t, and $h(t)$ be the depth of the water at time t.

i. Find a formula for $V(t)$ in terms of $h(t)$. $V(t) =$ ________________________

ii. Find the differential equation satisfied by $V(t)$. Include initial conditions.

Solution: i) The formula is $V(t) = 64\pi h(t)$.

ii) The differential equation is

$$\frac{dV}{dt} = 2 - kh.$$

So now we can solve $h(t) = \frac{V(t)}{64\pi}$. Substituting in V for h, we get

$$\frac{dV}{dt} = 2 - k\frac{V}{64\pi}$$

with initial condition $V(0) = 0$.

b. [7 points] Let $M(t)$ be the balance in dollars in a bank account t years after the initial deposit. The function $M(t)$ satisfies the differential equation

$$\frac{dM}{dt} = \frac{1}{100}M - a.$$

where a is a positive constant. Find a formula for $M(t)$ if the initial deposit is $1,000$ dollars. Your answer may depend on a.

Solution: This equation is separable:

$$\frac{dM}{M - 100a} = \frac{1}{100}dt.$$

Integrating, we find $\ln|M - 100a| = \frac{t}{100} + C$. So we get

$$M = Be^{t/100} + 100a.$$

Using the initial conditions, $M(0) = 1000$, so $1000 = B + 100a$. Substituting back in we get

$$M = 100\left((10 - a)e^{t/100} + a\right).$$